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How to Extend the Concept of Convexity Cuts to Derive Deeper Cutting Planes

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Abstract. A new type of cutting plane, termed a decomposition cut, is introduced that can be constructed under the same assumptions as the well-known convexity cut. Therefore it can be applied in algorithms (e.g. cutting plane, branch-and-cut) for various problems of global optimization, such as concave minimization, bilinear programming, reverse-convex programming, and integer programming. In computational tests with cutting plane algorithms for concave minimization, decomposition cuts were shown to be superior to convexity cuts.

Key words: Concave minimization, Concavity cut, Convexity cut, Cutting plane, Tuy cut

1. Introduction

In this paper we are concerned with cutting planes for optimization problems that are given in the form

$$\min\{\varphi(x) \mid x \in \mathrm{P} \cap \Omega\}, \tag{1}$$

where $\mathrm{P} \subseteq \mathbf{R}^n$ is a polyhedron, $\Omega \subseteq \mathbf{R}^n$ a set and $\varphi : \mathrm{P} \cap \Omega \mapsto \mathbf{R}$. This includes a wide range of optimization problems, such as concave minimization, bilinear programming, reverse-convex programming and integer programming. The integer program $\min\{c^T x \mid Ax \leqslant b, x \in \mathbf{I}^n\}$, for example, can be transformed into (1) by defining $\varphi(x) = c^T x$, $\mathrm{P} = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ and $\Omega = \mathbf{I}^n$.

A cutting plane $h^T x \geqslant \theta$ that reduces P without eliminating a point in $\mathrm{P} \cap \Omega$ is called a (P, Ω)-*cut*. For integer programming the Gomory cut is a well-known (P, Ω)-cut. A Gomory cut eliminates a nonintegral vertex of P without eliminating an integral solution, i.e. it reduces P but not $\mathrm{P} \cap \Omega$.

A more general class of (P, Ω)-cuts is the class of *convexity cuts*, introduced by Tuy [20] and extended by Glover [6, 7]. Convexity cuts have been used, for example, in concave minimization [4, 13, 14, 20, 21], bilinear programming [12, 15, 19, 22], reverse-convex programming [8, 9, 18] and integer programming [1–3, 23].

Let us suppose that the polyhedron P is full-dimensional and given in the form $\mathrm{P} = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$, where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$ and $m > n$. Let $x_0 \notin \mathrm{P} \cap \Omega$ be a vertex of P that is to be eliminated. A convexity cut $c^T(x - x_0) \geqslant 1$ is derived as follows.

First we construct a convex set K such that $x_0 \in \text{int(K)}$ and $\text{int(K)} \cap (\text{P} \cap \Omega) = \emptyset$. How this can be done for several different types of optimization problems has been described in detail elsewhere.

Next we derive a P-containing cone $\text{C}(x_0)$ as follows. Since x_0 is a vertex of P, there exists an $(n, n+1)$-submatrix (A_0, b_0) of full rank of (A, b) such that $A_0 x_0 = b_0$. By defining $\text{C}(x_0) = \{x \in \mathbf{R}^n \mid A_0 x \leqslant b_0\}$ we have $\text{P} \subseteq \text{C}(x_0)$. x_0 is the only vertex of $\text{C}(x_0)$ and there are n edges of $\text{C}(x_0)$ emanating from x_0, all of which are unbounded. Now let $u_1, u_2, \dots, u_n$ denote the directions of these edges. $u_1, u_2, \dots, u_n$ are linearly independent and we have $\text{C}(x_0) = x_0 + \text{cone}(u_1, u_2, \dots, u_n)$.

Then we determine $\hat{\tau}_k$ such that $x_0 + \hat{\tau}_k u_k$ is the intersection point of the cone edge $\text{E}_k = \{y_k(\tau) = x_0 + \tau u_k \mid \tau \in \mathbf{R}_0^+\}$ and the boundary of cl(K) if such an intersection point exists. If such a point does not exist, i.e. $\text{E}_k \subseteq \text{int(K)}$, we set $\hat{\tau}_k = \infty$. In a final step we choose τ_k with $0 < \tau_k \leqslant \hat{\tau}_k$ and determine the hyperplane $c^T(x - x_0) = 1$ that intersects the cone edge E_k at $y_i(\tau_k)$ if $\tau_k < \infty$ and is parallel to E_k if $\tau_k = \infty$, i.e. $c^T = \left(\frac{1}{\tau_1}, \frac{1}{\tau_2}, \dots, \frac{1}{\tau_n}\right) Q^{-1}$ with $Q = (u_1, u_2, \dots, u_n)$ and $\frac{1}{\tau_k} := 0$ for $\tau_k = \infty$.

Since K is convex, with x_0 the convexity cut $c^T(x - x_0) \geqslant 1$ excludes only points in the portion of $\text{C}(x_0)$ contained in int(K). We have $\text{P} \cap \Omega \subseteq \text{P} \subseteq \text{C}(x_0)$ and $\text{int(K)} \cap (\text{P} \cap \Omega) = \emptyset$. Thus $c^T(x - x_0) \geqslant 1$ excludes x_0 without excluding any point in $\text{P} \cap \Omega$, i.e. $c^T(x - x_0) \geqslant 1$ is a (P, Ω)-cut. The deepest convexity cut, called an *intersection cut*, is the convexity cut with $\tau_k = \hat{\tau}_k$ (see Figure 1(a)). Intersection cuts are also known as concavity cuts or as Tuy cuts.

The idea behind the convexity cut is to approximate the polyhedron P by the cone $\text{C}(x_0)$ and to eliminate only points in the portion of $\text{C}(x_0)$ contained in int(K). A problem with this cut is that the cone $\text{C}(x_0)$ is, in general, a poor approximation of the polyhedron P (cf. Zwart [24]). Hence the derived convexity cut may eliminate a large portion of $\text{C}(x_0) \cap \text{int(K)}$, but only a small portion of $\text{P} \cap \text{int(K)}$.

To overcome this problem we decompose the cone $\text{C}(x_0)$ into 2^t suitable cones that are of dimension $n - t$, where t with $1 \leqslant t \leqslant n$ denotes the respective level of decomposition such that the convex hull of these cones contains P. Using these cones we can derive a cutting plane, called a *decomposition cut*, which is related to convexity cuts but eliminates a much larger portion of $\text{P} \cap \text{int(K)}$ (see Figure 1(b)).

In computational tests with cutting plane algorithms for concave minimization, decomposition cuts were shown to be superior to intersection cuts. Some problems could be solved as much as 80 times faster with decomposition cuts than with intersection cuts.

The structure of this paper is as follows. First pseudovertices and cones derived with respect to (w.r.t.) pseudovertices are introduced. Then these concepts are applied to approximate polyhedra by cones. Next we discuss the decomposition

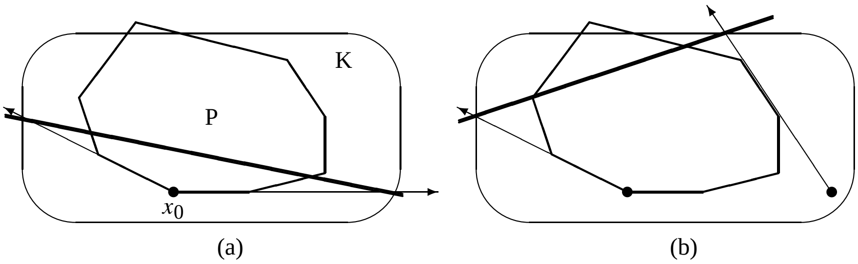

Figure 1. Intersection cut and decomposition cut.

of cones into cones of lower dimension. Then the procedure for deriving decomposition cuts is described. The paper concludes with a brief report on numerical experiments.

2. Pseudovertices and Cones

A vertex x_0 of the polyhedron $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ is a 0-dimensional face of P. This is equivalent to the conditions that $Ax_0 \leqslant b$ holds and that there exists an $(n, n+1)$-submatrix (A_0, b_0) of full rank of (A, b) such that $A_0 x_0 = b_0$, i.e. $x_0 = A_0^{-1} b_0$. By dropping the first condition we can extend this notion to a more general one, as in the following definition.

DEFINITION 2.1. Let $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ be a polyhedron with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$ and $\dim(P) = n$, and let $Ax \leqslant b$ include no constraints $a_i^T x \leqslant \beta_i$, $a_j^T x \leqslant \beta_j$ with $(a_i^T, \beta_i) = \lambda(a_j^T, \beta_j)$ for some $\lambda \in \mathbf{R}^+$.

1. Let $(\tilde{A}, \tilde{b})$ be an $(n, n+1)$-submatrix of full rank of (A, b), and let $\tilde{x}$ be the unique solution of $\tilde{A}x = \tilde{b}$. $\tilde{x}$ is called a pseudovertex of P, and the set of pseudovertices of P is denoted by $V^{ps}(P_{(A,b)})$.
2. If for $\tilde{x} \in V^{ps}(P_{(A,b)})$ there exists one and only one $(n, n+1)$-submatrix $(\tilde{A}, \tilde{b})$ of full rank of (A, b) such that $\tilde{A}\tilde{x} = \tilde{b}$, then $\tilde{x}$ is called a nondegenerate pseudovertex. Otherwise $\tilde{x}$ is a degenerate pseudovertex.
3. If for $\tilde{x}_1, \tilde{x}_2 \in V^{ps}(P_{(A,b)})$ there exist $(n, n+1)$-submatrices $(\tilde{A}_1, \tilde{b}_1)$, $(\tilde{A}_2, \tilde{b}_2)$ of full rank of (A, b) such that $\tilde{A}_1\tilde{x}_1 = \tilde{b}_1$, $\tilde{A}_2\tilde{x}_2 = \tilde{b}_2$, and $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ differ in exactly one row, then $\tilde{x}_1, \tilde{x}_2$ are neighbors.

A pseudovertex of P is a vertex if it belongs to P. For a vertex x_0 of P there exists at least one $(n, n+1)$-submatrix (A_0, b_0) of full rank of (A, b) such that $A_0 x_0 = b_0$. If there exists only one such submatrix, then x_0 is nondegenerate. Otherwise x_0 is degenerate. This observation leads to the definition of nondegeneracy and degeneracy of pseudovertices.

The definition of neighborhood for pseudovertices is an extension of the usual definition of neighborhood for vertices. In fact, vertices x_1, x_2 of P that are connected by an edge are neighbors, i.e. there exist $(n, n+1)$-submatrices (A_1, b_1),

(A_2, b_2) of full rank of (A, b) such that $A_1x_1 = b_1$, $A_2x_2 = b_2$, and (A_1, b_1) and (A_2, b_2) differ in one row.

We now describe three types of neighborhood for pseudovertices. Let $\tilde{x}_1, \tilde{x}_2 \in \mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ be neighbors, and let $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ differ only in the last row, i.e. there exist $(\breve{A}, \breve{b}) \in \mathbf{R}^{n-1\times n+1}$, $(\tilde{a}_{1,n}, \tilde{\beta}_{1,n})$, $(\tilde{a}_{2,n}, \tilde{\beta}_{2,n}) \in \mathbf{R}^{n+1}$ with $(\tilde{a}_{1,n}, \tilde{\beta}_{1,n}) \neq (\tilde{a}_{2,n}, \tilde{\beta}_{2,n})$, such that

$$(\tilde{A}_i, \tilde{b}_i) = \begin{pmatrix} \breve{A}, & \breve{b} \\ \tilde{a}_{i,n}^T, & \tilde{\beta}_{i,n} \end{pmatrix} \quad \text{for } i = 1, 2. \tag{2}$$

Consider

$$\mathrm{G} = \{x \in \mathbf{R}^n \mid \breve{A}x = \breve{b}\}. \tag{3}$$

The set G is a line that is intersected by the hyperplanes $\tilde{a}_{1,n}^T x = \tilde{\beta}_{1,n}$ and $\tilde{a}_{2,n}^T x = \tilde{\beta}_{2,n}$. The intersection of G with $\tilde{a}_{1,n}^T x = \tilde{\beta}_{1,n}$ defines the pseudovertex $\tilde{x}_1$, and the intersection of G with $\tilde{a}_{2,n}^T x = \tilde{\beta}_{2,n}$ defines the pseudovertex $\tilde{x}_2$. Now we consider the half-lines

$$\mathrm{G}_1 = \{x \in \mathrm{G} \mid \tilde{a}_{1,n}^T x \leqslant \tilde{\beta}_{1,n}\} \text{ and } \mathrm{G}_2 = \{x \in \mathrm{G} \mid \tilde{a}_{2,n}^T x \leqslant \tilde{\beta}_{2,n}\}, \tag{4}$$

which originate at $\tilde{x}_1$ and $\tilde{x}_2$, respectively, and are contained in G. There are three possible cases, whether $\tilde{x}_2 \in \mathrm{G}_1$ or $\tilde{x}_1 \in \mathrm{G}_2$, or both. This leads to the following types of neighborhood:

- N_1-neighborhood: $\tilde{x}_1, \tilde{x}_2 \in \mathrm{G}_1 \cap \mathrm{G}_2$;
- N_2-neighborhood: $\tilde{x}_1 \in \mathrm{G}_1 \cap \mathrm{G}_2 \ \wedge\ \tilde{x}_2 \notin \mathrm{G}_1 \cap \mathrm{G}_2$ or
  $\tilde{x}_1 \notin \mathrm{G}_1 \cap \mathrm{G}_2 \ \wedge\ \tilde{x}_2 \in \mathrm{G}_1 \cap \mathrm{G}_2$;
- N_3-neighborhood: $\tilde{x}_1, \tilde{x}_2 \notin \mathrm{G}_1 \cap \mathrm{G}_2$.

The N_1, N_2, N_3 neighborhood concepts are equivalent to $\mathrm{G}_1 \cap \mathrm{G}_2$ being nonempty and bounded, unbounded, and empty, respectively.

EXAMPLE 2.1. Let the polyhedron P of Figure 2 be described by $\mathrm{P} = \{x \in \mathbf{R}^3 \mid a_1^T x \leqslant \beta_1, a_2^T x \leqslant \beta_2, \dots, a_6^T x \leqslant \beta_6\}$. In Figure 2(a) the facets $\{x \in \mathrm{P} \mid a_i^T x = \beta_i\}$ of P are denoted by F_i. In Figure 2(b) the pseudovertices of P are indicated by dots. For example, the intersection point of the hyperplanes $a_3^T x = \beta_3$, $a_4^T x = \beta_4$, $a_5^T x = \beta_5$ defines the pseudovertex $\tilde{x}_1$, and the intersection point of the hyperplanes $a_1^T x = \beta_1$, $a_4^T x = \beta_4$, $a_5^T x = \beta_5$ defines the pseudovertex $\tilde{x}_2$. According to Definition 2.1.3, $\tilde{x}_1$ and $\tilde{x}_2$ are neighbors. $\tilde{x}_1$ and $\tilde{x}_2$ lie on the line $\mathrm{G} = \{x \in \mathbf{R} \mid a_4^T x = \beta_4, a_5^T x = \beta_5\}$ (see Figure 2(b)). G_1 and G_2 are defined by $\mathrm{G}_1 = \{x \in \mathrm{G} \mid a_3^T x \leqslant \beta_3\}$ and $\mathrm{G}_2 = \{x \in \mathrm{G} \mid a_1^T x \leqslant \beta_1\}$. We have $a_1^T \tilde{x}_1 > \beta_1$ and $a_3^T \tilde{x}_2 > \beta_3$. Therefore we have $\mathrm{G}_1 \cap \mathrm{G}_2 = \emptyset$, and $\tilde{x}_1$ and $\tilde{x}_2$ are N_3-neighbors.

All pairs of pseudovertices lying on one of the lines indicated in Figure 2(b) are neighbors, e.g. $\tilde{x}_2$ and $\tilde{x}_3$ are neighbors, as are $\tilde{x}_3$ and $\tilde{x}_4$. Similarly, we can verify that $\tilde{x}_2$ and $\tilde{x}_3$ are N_2-neighbors, and that $\tilde{x}_3$ and $\tilde{x}_4$ are N_1-neighbors.

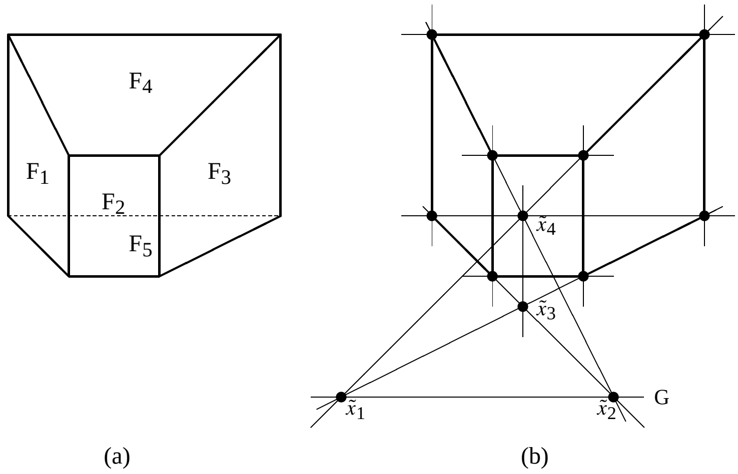

Figure 2. A polyhedron and its pseudovertices.

DEFINITION 2.2. Let $\tilde{x} \in V^{ps}(P_{(A,b)})$ be nondegenerate with $\tilde{A}\tilde{x} = \tilde{b}$, where $(\tilde{A}, \tilde{b})$ is an $(n, n+1)$-submatrix of full rank of (A, b).

1. The cone $C(\tilde{x})$ derived w.r.t. the pseudovertex $\tilde{x}$ is defined by

$$\begin{aligned} C(\tilde{x}) &= \{x \in \mathbf{R}^n \mid \tilde{A}x \leqslant \tilde{b}\} \\ &= \tilde{x} + \text{cone}(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n), \end{aligned}$$

   where $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n$ are directions of the edges of $C(\tilde{x})$.
2. A set $S \subseteq V^{ps}(P_{(A,b)})$ of nondegenerate pseudovertices containing no N_2-neighbors is called an N-set of $V^{ps}(P_{(A,b)})$. For $\tilde{x} \in S$ we denote by $C_S(\tilde{x})$ the face of $C(\tilde{x})$ which is spanned by the vectors $\tilde{u}_k$ such that the edge $E_k = \{\tilde{x} + \lambda\tilde{u}_k \mid \lambda \in \mathbf{R}_0^+\}$ and its negative extension $E_k^- = \{\tilde{x} + \lambda\tilde{u}_k \mid \lambda \in \mathbf{R}_0^-\}$ contain no pseudovertex in $S\backslash\{\tilde{x}\}$.

In Definition 2.2 only cones derived with respect to nondegenerate pseudovertices are considered. If a pseudovertex $\tilde{x} \in V^{ps}(P_{(A,b)})$ is degenerate, there are several ways to deal with this. One is to make all pseudovertices of $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ nondegenerate by slightly perturbing the vector b. A second is the following, which is adopted from Balas [1]. Since $\tilde{x}$ is a pseudovertex, among the constraints that define P we can always find n linearly independent constraints that are binding for $\tilde{x}$. Let P' denote the polyhedron obtained from P by omitting all the other binding constraints for $\tilde{x}$. Then we have $P \subseteq P'$, and $\tilde{x}$ is a nondegenerate pseudovertex of P'. Hence we can derive the cones $C(\tilde{x})$ and $C_S(\tilde{x})$ w.r.t. P'.

COROLLARY 2.1. *Let* $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ *be a polyhedron with* $\dim(P) = n$, *and let* $S = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$ *be an N-set of* $V^{ps}(P_{(A,b)})$. *Let* $A_S x \leqslant b_S$ *denote the system obtained from* $Ax \leqslant b$ *by omitting all constraints that are not binding*

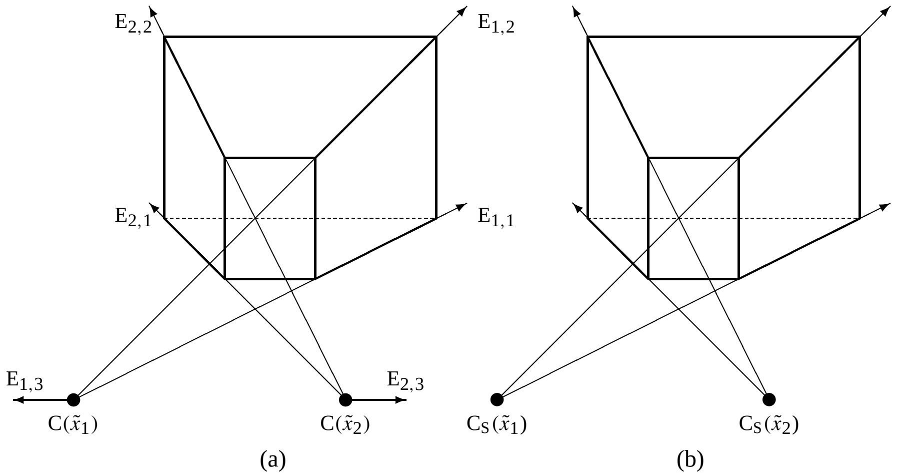

Figure 3. Cones derived with respect to pseudovertices.

for at least one pseudovertex in S*, and let* $\mathrm{P_S} = \{x \in \mathbf{R}^n \mid A_{\mathrm{S}}x \leqslant b_{\mathrm{S}}\}$ *be the corresponding polyhedron.*

Then we have $\mathrm{P} \subseteq \mathrm{P_S}$*, and* S *is also an* N*-set of* $\mathrm{V}^{ps}(\mathrm{P}_{\mathrm{S}_{(A_{\mathrm{S}},b_{\mathrm{S}})}})$*. For* $\tilde{x}_i \in \mathrm{S}$ *the cones* $\mathrm{C}(\tilde{x}_i)$ *and* $\mathrm{C_S}(\tilde{x}_i)$ *derived w.r.t.* $\mathrm{P_S}$ *are identical with the corresponding cones derived w.r.t.* P.

Proof. The system $A_{\mathrm{S}}x \leqslant b_{\mathrm{S}}$ that describes $\mathrm{P_S}$ is a subsystem of the system $Ax \leqslant b$ that describes P. Hence, we have $\mathrm{P} \subseteq \mathrm{P_S}$.

Since S is an N-set of $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$, each pseudovertex in S is nondegenerate. Thus, for $\tilde{x}_i \in \mathrm{S}$ there exists one and only one $(n, n+1)$-submatrix $(\tilde{A}_i, \tilde{b}_i)$ of full rank of (A, b) such that $\tilde{A}_i\tilde{x}_i = \tilde{b}_i$. Hence, $(\tilde{A}_i, \tilde{b}_i)$ is also the only submatrix of full rank of $(A_{\mathrm{S}}, b_{\mathrm{S}})$ such that $\tilde{A}_i\tilde{x}_i = \tilde{b}_i$. Therefore, $\tilde{x}_i$ is a nondegenerate pseudovertex of $\mathrm{P_S}$, and the cone $\mathrm{C}(\tilde{x}_i) = \{x \in \mathbf{R}^n \mid \tilde{A}_i x \leqslant \tilde{b}_i\}$ derived w.r.t. $\mathrm{P_S}$ is identical with the cone derived w.r.t. P.

Since for $\tilde{x}_i, \tilde{x}_j \in \mathrm{S}$ the corresponding $(n, n+1)$-submatrices of full rank of (A, b) and $(A_{\mathrm{S}}, b_{\mathrm{S}})$ are identical, the neighborhood relations for pseudovertices in S remain the same in $\mathrm{P_S}$ as in P. Thus S is also an N-set of $\mathrm{V}^{ps}(\mathrm{P}_{\mathrm{S}_{(A_{\mathrm{S}},b_{\mathrm{S}})}})$ and the cones $\mathrm{C_S}(\tilde{x}_i)$ and $\mathrm{C_S}(\tilde{x}_j)$ derived w.r.t. $\mathrm{P_S}$ are identical with the corresponding cones derived w.r.t. P. □

EXAMPLE 2.2. The cones derived w.r.t. $\tilde{x}_1, \tilde{x}_2 \in \mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ are

$$\mathrm{C}(\tilde{x}_1) = \{x \in \mathbf{R}^3 \mid a_3^T x \leqslant \beta_3, a_4^T x \leqslant \beta_4, a_5^T x \leqslant \beta_5\}$$

with $\mathrm{C}(\tilde{x}_1) = \tilde{x}_1 + \mathrm{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}, \tilde{u}_{1,3})$, and

$$\mathrm{C}(\tilde{x}_2) = \{x \in \mathbf{R}^3 \mid a_1^T x \leqslant \beta_1, a_4^T x \leqslant \beta_4, a_5^T x \leqslant \beta_5\}$$

with $C(\tilde{x}_1) = \tilde{x}_1 + \text{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}, \tilde{u}_{1,3})$ (see Figure 3(a)). Since $\tilde{x}_1$ and $\tilde{x}_2$ are N_3-neighbors, the set $S = \{\tilde{x}_1, \tilde{x}_2\}$ is an N-set of $V^{ps}(P_{(A,b)})$. Let $E_{i,j}$ denote the cone edge $\{\tilde{x}_i + \lambda\tilde{u}_{i,j} \mid \lambda \in \mathbf{R}_0^+\}$ and let $E_{i,j}^-$ be its negative extension, i.e. $E_{i,j}^- = \{\tilde{x}_i + \lambda\tilde{u}_{i,j} \mid \lambda \in \mathbf{R}_0^-\}$. We have $\tilde{x}_2 \in E_{1,3}^-$ and $\tilde{x}_1 \in E_{2,3}^-$. The other edges of $C(\tilde{x}_1)$ and $C(\tilde{x}_2)$, and their negative extensions, contain no pseudovertex in $S\setminus\{\tilde{x}_1\}$ and $S\setminus\{\tilde{x}_2\}$, respectively. Hence, we have $C_S(\tilde{x}_1) = \tilde{x}_1 + \text{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2})$ and $C_S(\tilde{x}_2) = \tilde{x}_2 + \text{cone}(\tilde{u}_{2,1}, \tilde{u}_{2,2})$ (see Figure 3(b)).

3. Approximation of Polyhedra by Cones

For a pseudovertex $\tilde{x}_1 \in V^{ps}(P_{(A,b)})$ the corresponding cone $C(\tilde{x}_1)$ contains the polyhedron $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$, i.e. $P \subseteq C(\tilde{x}_1)$. The idea is to choose an N-set $S = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$ and to replace the cone $C(\tilde{x}_1)$ by the collection of cones $C_S(\tilde{x}_1), C_S(\tilde{x}_2), \ldots, C_S(\tilde{x}_l)$. We shall now verify by Theorem 3.1 that the convex hull of these cones contains P. Therefore $\text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$ provides an approximation of the polyhedron P. The following corollary will be helpful in proving Theorem 3.1 and can be proved itself by applying concepts described, for instance, by Schrijver [17], Chapter 8.

COROLLARY 3.1. *Let* P *be a pointed polyhedron with* $\dim(P) \geqslant 2$, *and let* $F_1, F_2, \ldots, F_h$ *be the facets of* P. *Then we have* $P = \text{conv}(\bigcup_{j=1}^{h} F_j)$.

THEOREM 3.1. *Let* $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ *be a pointed polyhedron with* $\dim(P) = n \geqslant 2$, *and let* $S \neq \emptyset$ *be an N-set of* $V^{ps}(P_{(A,b)})$. *Then we have* $P \subseteq \text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$.

Proof. The idea behind the proof is the following. We consider the P-containing polyhedron P_S (cf. Corollary 2.1) and prove that each of its facets is contained in $\text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$. Hence, the convex hull of the facets of P_S is also contained in $\text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$. However, it follows from the definition of P_S that P_S fulfills the conditions of Corollary 3.1. Therefore, the convex hull of its facets contains P_S itself. Hence, we have $P \subseteq P_S \subseteq \text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$, which proves the theorem. Therefore, we only have to verify that each facet of P_S is contained in $\text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$. We prove this by induction in n.

<u>n=2:</u> Suppose that $P = \{x \in \mathbf{R}^2 \mid Ax \leqslant b\}$ with $\dim(P) = 2$ is a pointed polyhedron and let $S \neq \emptyset$ be an N-set of $V^{ps}(P_{(A,b)})$. Let F_{j_1} be a facet of P_S, i.e. $\dim(F_{j_1}) = 1$ and there exists a constraint $a_{j_1}^T x \leqslant \beta_{j_1}$ of $A_S x \leqslant b_S$ such that $F_{j_1} = \{x \in P_S \mid a_{j_1}^T x = \beta_{j_1}\}$. It follows from the definition of P_S that there exists $\tilde{x}_i \in S$ such that $a_{j_1}^T \tilde{x}_i = \beta_{j_1}$. Since $\tilde{x}_i$ is a nondegenerate pseudovertex, there exists exactly one more constraint $a_{j_2}^T x \leqslant \beta_{j_2}$ of $A_S x \leqslant b_S$ such that $a_{j_2}^T \tilde{x}_i = \beta_{j_2}$, and $a_{j_1}^T x = \beta_{j_1}$ and $a_{j_2}^T x = \beta_{j_2}$ are linearly independent. Thus, we have

$$\begin{aligned} C(\tilde{x}_i) &= \{x \in \mathbf{R}^2 \mid a_{j_1}^T x \leqslant \beta_{j_1}, a_{j_2}^T x \leqslant \beta_{j_2}\} \\ &= \tilde{x}_i + \text{cone}(\tilde{u}_{i,1}, \tilde{u}_{i,2}), \end{aligned} \tag{5}$$

where $\tilde{u}_{i,1}, \tilde{u}_{i,2} \in \mathbf{R}^2$ are directions of the edges of $C(\tilde{x}_i)$, i.e. $E_{i,1} = \{\tilde{x}_i + \lambda\tilde{u}_{i,1} \mid \lambda \in \mathbf{R}_0^+\}$ and $E_{i,2} = \{\tilde{x}_i + \lambda\tilde{u}_{i,2} \mid \lambda \in \mathbf{R}_0^+\}$, where

$$
\begin{aligned}
&E_{i,1} = \{x \in \mathbf{R}^2 \mid a_{j_1}^T x \leqslant \beta_{j_1}, a_{j_2}^T x = \beta_{j_2}\} \text{ and} \\
&E_{i,2} = \{x \in \mathbf{R}^2 \mid a_{j_1}^T x = \beta_{j_1}, a_{j_2}^T x \leqslant \beta_{j_2}\}.
\end{aligned} \tag{6}
$$

Since $F_{j_1} = \{x \in P_S \mid a_{j_1}^T x = \beta_{j_1}\}$ is a facet of P_S and $a_{j_2}^T x \leqslant \beta_{j_2}$ is a P_S-describing inequality, we have $F_{j_1} \subseteq E_{i,2}$. Note that $E_{i,2}$ is not necessarily an edge of the cone $C_S(\tilde{x}_i)$.

Case 1: Suppose that $E_{i,2}$ is an edge of $C_S(\tilde{x}_i)$. Then we have $F_{j_1} \subseteq E_{i,2} \subseteq C_S(\tilde{x}_i)$, which verifies that $F_{j_1} \subseteq \operatorname{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$.

Case 2: Suppose that $E_{i,2}$ is not an edge of $C_S(\tilde{x}_i)$. Then it follows from the definition of the cone $C_S(\tilde{x}_i)$ that there exists $\tilde{x}_l \in S$ with $\tilde{x}_l \in E_{i,2} \cup E_{i,2}^- \setminus \{\tilde{x}_i\}$, where $E_{i,2}^-$ denotes the negative extension of $E_{i,2}$. This implies $a_{j_1}^T \tilde{x}_l = \beta_{j_1}$ (see (6)), i.e. $\tilde{x}_i$ and $\tilde{x}_l$ are neighbors.

Since $\tilde{x}_l$ is nondegenerate, there exists exactly one more constraint $a_{j_3}^T x \leqslant \beta_{j_3}$ of $A_S x \leqslant b_S$ such that $a_{j_3}^T \tilde{x}_l = \beta_{j_3}$, and $a_{j_1}^T x = \beta_{j_1}$ and $a_{j_3}^T x = \beta_{j_3}$ are linearly independent. For the neighbors $\tilde{x}_i$ and $\tilde{x}_l$ let us consider the line $G = \{x \in \mathbf{R}^2 \mid a_{j_1}^T x = \beta_{j_1}\}$ (see (3)) and the half-lines

$$
\begin{aligned}
&G_1 = \{x \in \mathbf{R}^2 \mid a_{j_1}^T x = \beta_{j_1},\ a_{j_2}^T x \leqslant \beta_{j_2}\} \text{ and} \\
&G_2 = \{x \in \mathbf{R}^2 \mid a_{j_1}^T x = \beta_{j_1},\ a_{j_3}^T x \leqslant \beta_{j_3}\}
\end{aligned}
$$

(see (4)). We have $F_{j_1} = \{x \in P_S \mid a_{j_1}^T x = \beta_{j_1}\}$, and $a_{j_2}^T x \leqslant \beta_{j_2}$ and $a_{j_3}^T x \leqslant \beta_{j_3}$ are P_S describing constraints. This implies

$$
F_{j_1} \subseteq G_1 \cap G_2. \tag{7}
$$

Since S is an N-set, the neighbors $\tilde{x}_i, \tilde{x}_l \in S$ have to be N_1- or N_3-neighbors. We claim that $\tilde{x}_i$ and $\tilde{x}_l$ are N_1-neighbors, which they are, since if we assume that $\tilde{x}_i$ and $\tilde{x}_l$ are N_3-neighbors, then we have $G_1 \cap G_2 = \emptyset$ and by (7) we have $F_{j_1} = \emptyset$, which contradicts $\dim(F_{j_1}) = 1$. Since $\tilde{x}_i$ and $\tilde{x}_l$ are N_1-neighbors, $G_1 \cap G_2$ is bounded, and we have $G_1 \cap G_2 = \operatorname{conv}(\tilde{x}_i, \tilde{x}_l)$. Hence, by (7) we have

$$
F_{j_1} \subseteq \operatorname{conv}(\tilde{x}_i, \tilde{x}_l) \subseteq \operatorname{conv}(C_S(\tilde{x}_i), C_S(\tilde{x}_l)), \tag{8}
$$

which verifies that $F_{j_1} \subseteq \operatorname{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i))$ for Case 2.

F_{j_1} is an arbitrary facet of P_S. Based on the considerations at the beginning of the proof this proves Theorem 3.1 for $n = 2$.

$\underline{n-1 \to n}$: Let Theorem 3.1 hold for all full-dimensional and pointed polyhedra in $\mathbf{R}^k$ with $2 \leqslant k \leqslant n-1$. Suppose that $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ is a pointed polyhedron with $\dim(P) = n \geqslant 3$, and let S be an N-set of $V^{ps}(P_{(A,b)})$. Let F_{j_1} be a facet of the P-containing polyhedron $P_S = \{x \in \mathbf{R}^n \mid A_S x \leqslant b_S\}$, and let $a_{j_1}^T x \leqslant \beta_{j_1}$ be the corresponding constraint of $A_S x \leqslant b_S$ such that $F_{j_1} = \{x \in P_S \mid a_{j_1}^T x = \beta_{j_1}\}$.

We define $S_{j_1} := S \cap \mathrm{aff}(F_{j_1})$, where $\mathrm{aff}(F_{j_1})$ is the affine hull of F_{j_1}, i.e. $\mathrm{aff}(F_{j_1}) = \{x \in \mathbf{R}^n \mid a_{j_1}^T x = \beta_{j_1}\}$. It follows from its definition that P_S is a pointed polyhedron with $\dim(P_S) = n$, and that S_{j_1} is nonempty. Furthermore, because of $S_{j_1} \subseteq S$ the set S_{j_1} is also an N-set.

Since P_S is a pointed polyhedron with $\dim(P_S) = n$, its facet F_{j_1} is also a pointed polyhedron with $\dim(F_{j_1}) = n-1$. To apply the induction hypothesis we have to map F_{j_1} into $\mathbf{R}^{n-1}$. To do this we choose $\tilde{x}_{j_1} \in S_{j_1}$ and a basis $\{v_1, v_2, \dots, v_{n-1}\}$ of the linear space $\mathrm{aff}(F_{j_1}) - \tilde{x}_{j_1} = \{v \in \mathbf{R}^n \mid v + \tilde{x}_{j_1} \in \mathrm{aff}(F_{j_1})\}$ and define with $V_{j_1} = (v_1, v_2, \dots, v_{n-1}) \in \mathbf{R}^{n \times n-1}$ the mappings

$$\begin{aligned} &\phi_{j_1} : \mathrm{aff}(F_{j_1}) \mapsto \mathbf{R}^{n-1} \text{with} \phi_{j_1}(x) = (V_{j_1}^T V_{j_1})^{-1} V_{j_1}^T (x - \tilde{x}_{j_1}); \\ &\phi_{j_1}^{-1} : \mathbf{R}^{n-1} \mapsto \mathrm{aff}(F_{j_1}) \text{with} \phi_{j_1}^{-1}(y) = V_{j_1} y + \tilde{x}_{j_1}. \end{aligned} \tag{9}$$

$\phi_{j_1} : \mathrm{aff}(F_{j_1}) \mapsto \mathbf{R}^{n-1}$ sets up a one-to-one correspondence between $\mathrm{aff}(F_{j_1})$ and $\mathbf{R}^{n-1}$, and $\phi_{j_1}^{-1} : \mathbf{R}^{n-1} \mapsto \mathrm{aff}(F_{j_1})$ is its inverse. We have

$$\phi_{j_1}(F_{j_1}) = \{y \in \mathbf{R}^{n-1} \mid A_S(V_{j_1} y + \tilde{x}_{j_1}) \leqslant b_S, a_{j_1}^T (V_{j_1} y + \tilde{x}_{j_1}) = \beta_{j_1}\}. \tag{10}$$

$\phi_{j_1}(F_{j_1})$ is a full-dimensional and pointed polyhedron in $\mathbf{R}^{n-1}$, and $\phi_{j_1}(S_{j_1}) \neq \emptyset$ is an N-set of $\phi_{j_1}(F_{j_1})$. Thus, by defining $\tilde{y}_i := \phi_{j_1}(\tilde{x}_i)$ for $\tilde{x}_i \in S_{j_1}$ we get by the induction hypothesis

$$\phi_{j_1}(F_{j_1}) \subseteq \mathrm{conv}\left(\bigcup_{\tilde{y}_i \in \phi_{j_1}(S_{j_1})} \widehat{C}_{\phi_{j_1}(S_{j_1})}(\tilde{y}_i) \right), \tag{11}$$

where $\widehat{C}_{\phi_{j_1}(S_{j_1})}(\tilde{y}_i)$ denotes the respective cone derived in $\mathbf{R}^{n-1}$. Since $\phi_{j_1}^{-1} : \mathbf{R}^{n-1} \mapsto \mathrm{aff}(F_{j_1})$ is affine and linear we have

$$\phi_{j_1}^{-1}\left(\mathrm{conv}\left(\bigcup_{\tilde{y}_i \in \phi_{j_1}(S_{j_1})} \widehat{C}_{\phi_{j_1}(S_{j_1})}(\tilde{y}_i) \right) \right) = \mathrm{conv}\left(\bigcup_{\tilde{x}_i \in S_{j_1}} \phi_{j_1}^{-1}\left(\widehat{C}_{\phi_{j_1}(S_{j_1})}(\phi_{j_1}(\tilde{x}_i)) \right) \right).$$

Furthermore, we have $\phi_{j_1}^{-1}\left(\widehat{C}_{\phi_{j_1}(S_{j_1})}(\phi_{j_1}(\tilde{x}_i)) \right) = C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})}$, where by $C_{S_{j_1}}(\tilde{x}_i)$ we denote the cone that is derived in $\mathbf{R}^n$ w.r.t. the polyhedron P_s and the N-set S_{j_1}, and by $C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})}$ we denote the cone $C_{S_{j_1}}(\tilde{x}_i) \cap \mathrm{aff}(F_{j_1})$. Therefore, by (11) we get

$$F_{j_1} \subseteq \mathrm{conv}\left(\bigcup_{\tilde{x}_i \in S_{j_1}} C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})} \right). \tag{12}$$

We have $C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})} = C_S(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})} \subseteq C_S(\tilde{x}_i)$ and $S_{j_1} \subseteq S$. Thus

$$F_{j_1} \subseteq \mathrm{conv}\left(\bigcup_{\tilde{x}_i \in S_{j_1}} C_S(\tilde{x}_i) \right) \subseteq \mathrm{conv}\left(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i) \right).$$

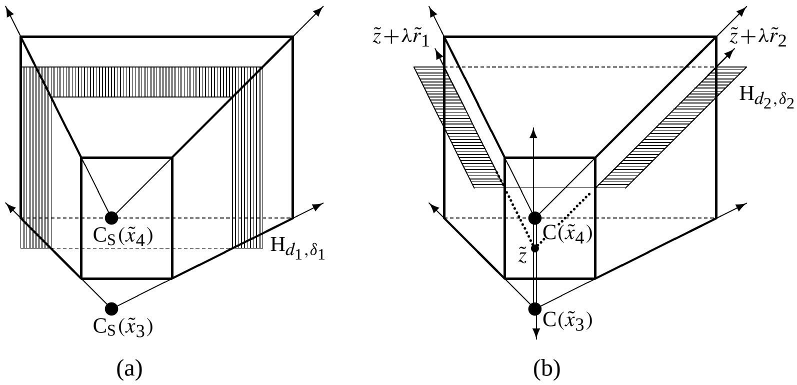

Figure 4. Cutting planes and approximation of the reduced polyhedron by cones.

F_{j_1} is an arbitrary facet of P_S. Based on the considerations at the beginning of the proof we have therefore proved Theorem 3.1. □

For an N-set S of $V^{ps}(P_{(A,b)})$ $\mathrm{conv}(\bigcup_{\tilde{x}_i\in S} C_S(\tilde{x}_i))$ provides an approximation of the polyhedron P. Our aim is to derive a (P, Ω)-cut. To show that a cutting plane $d^Tx \geqslant \delta$ is a (P, Ω)-cut, we have to verify that $P \cap \{x \in \mathbf{R}^n \mid d^Tx < \delta\}$ and Ω are disjunct. To do this we shall provide, by Theorem 3.2, a method that allows us to derive an approximation of the reduced polyhedron $P \cap \{x \in \mathbf{R}^n \mid d^Tx \leqslant \delta\}$ from the collection of cones $C_S(\tilde{x}_i)$, $\tilde{x}_i \in S$. To simplify notation, we hereafter denote by $H_{d,\delta}$, $H^{\oplus}_{d,\delta}$, $H^{+}_{d,\delta}$, $H^{\ominus}_{d,\delta}$ and $H^{-}_{d,\delta}$ the sets $\{x \in \mathbf{R}^n \mid d^Tx = \delta\}$, $\{x \in \mathbf{R}^n \mid d^Tx \geqslant \delta\}$, $\{x \in \mathbf{R}^n \mid d^Tx > \delta\}$, $\{x \in \mathbf{R}^n \mid d^Tx \leqslant \delta\}$, and $\{x \in \mathbf{R}^n \mid d^Tx < \delta\}$, respectively.

EXAMPLE 3.1. The pseudovertices $\tilde{x}_3$ and $\tilde{x}_4$ are N_1-neighbors (cf. Example 2.1). Since $S = \{\tilde{x}_3, \tilde{x}_4\}$ is an N-set of $V^{ps}(P_{(A,b)})$, we have $P \subseteq \mathrm{conv}(C_S(\tilde{x}_3), C_S(\tilde{x}_4))$. Consider the cutting plane $d_1^Tx \geqslant \delta_1$ with $d_1^T\tilde{x}_3 < \delta_1$ and $d_1^T\tilde{x}_4 < \delta_1$ in Figure 4(a). We shall verify by Theorem 3.2 that $P \cap H^{\ominus}_{d_1,\delta_1} \subseteq \mathrm{conv}(C_S(\tilde{x}_3) \cap H^{\ominus}_{d_1,\delta_1}, C_S(\tilde{x}_4) \cap H^{\ominus}_{d_1,\delta_1})$.

The situation is more complicated for the cutting plane $d_2^Tx \geqslant \delta_2$ with $d_2^T\tilde{x}_3 < \delta_2$ and $d_2^T\tilde{x}_4 > \delta_2$, which is indicated in Figure 4(b). We have $P \cap H^{\ominus}_{d_2,\delta_2} \not\subseteq \mathrm{conv}(C_S(\tilde{x}_3) \cap H^{\ominus}_{d_2,\delta_2}, C_S(\tilde{x}_4) \cap H^{\ominus}_{d_2,\delta_2})$. However, by Theorem 3.2 we shall verify $P \cap H^{\ominus}_{d_2,\delta_2} \subseteq \mathrm{conv}(C_S(\tilde{x}_3) \cap H^{\ominus}_{d_2,\delta_2}, \tilde{x}_4) + \mathrm{cone}(\tilde{r}_1, \tilde{r}_2)$, where $\tilde{r}_1, \tilde{r}_2 \in \mathbf{R}^3$ are directions of the half-lines that are defined by the intersection of H_{d_2,δ_2} with 2-dimensional faces of $C(\tilde{x}_3)$ and $C(\tilde{x}_4)$.

Theorem 3.2 will be proved similarly to Theorem 3.1. Hence, we need an analogue to Corollary 3.1.

COROLLARY 3.2. *Let* P *be a pointed polyhedron with* $\dim(P) \geqslant 2$, *let* $F_1, F_2, \ldots, F_h$ *be the facets of* P, *and let* $d^Tx \geqslant \delta$ *be a cutting plane* ($d \in \mathbf{R}^n \setminus \{0\}$,

$\delta \in \mathbf{R}$). *Then*

$$\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} = \mathrm{conv}\left(\bigcup_{j=1}^{h} \left[\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta} \right] \right) + \mathrm{cone}(r), \tag{13}$$

where r is the direction of the half-line $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ *if*

$$\emptyset \neq \mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} \neq \mathrm{P};$$
$$\dim(\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}) = \dim(\mathrm{P}) = 2;$$
$$\mathrm{P} \cap \mathrm{H}_{d,\delta} \textit{is unbounded},$$

and $r = 0$ otherwise.

Proof. Suppose that $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} = \emptyset$. Then we have $\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta} = \emptyset$ ($j = 1, 2, \ldots, h$), and Corollary 3.2 follows immediately. Suppose that $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} = \mathrm{P}$. Then we have $\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta} = \mathrm{F}_j$, and Corollary 3.2 follows from Corollary 3.1.

Therefore, let us suppose that $\emptyset \neq \mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} \neq \mathrm{P}$. Since P is pointed the polyhedra $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ and $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ are also pointed. We have to distinguish between $\dim(\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}) < \dim(\mathrm{P})$ and $\dim(\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}) = \dim(\mathrm{P})$.

Case 1: Suppose that $\dim(\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}) < \dim(\mathrm{P})$. Then $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ is a subset of a proper face of P. However, every face of P, except for P itself, is the intersection of facets of P. Hence, there exists a facet F_{j_0} of P such that $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} \subseteq \mathrm{F}_{j_0} \cap \mathrm{H}^{\ominus}_{d,\delta}$. Thus, because of $\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta} \subseteq \mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ for $j = 1, 2, \ldots, h$ we have (13).

Case 2: Suppose that $\dim(\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}) = \dim(\mathrm{P}) \geqslant 2$. The facets of the polyhedron $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ are subsets of the sets $\mathrm{P} \cap \mathrm{H}_{d,\delta}$, $\mathrm{F}_1 \cap \mathrm{H}^{\ominus}_{d,\delta}$, $\mathrm{F}_2 \cap \mathrm{H}^{\ominus}_{d,\delta}, \ldots, \mathrm{F}_h \cap \mathrm{H}^{\ominus}_{d,\delta}$. It follows from $\mathrm{P} \cap \mathrm{H}_{d,\delta} \subseteq \mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ and $\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta} \subseteq \mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ for $j = 1, 2, \ldots, h$, and from Corollary 3.1 that

$$\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} = \mathrm{conv}\left(\bigcup_{j=1}^{h} \left[\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta} \right], \mathrm{P} \cap \mathrm{H}_{d,\delta} \right). \tag{14}$$

Since $\emptyset \neq \mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta} \neq \mathrm{P}$, the set $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is a facet of $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$ (cf. Schrijver [17], Theorem 8.1). We verify (13) for Case 2 by considering the following cases.

(a) Let us suppose that $\dim(\mathrm{P} \cap \mathrm{H}_{d,\delta}) = \dim(\mathrm{P}) \geqslant 3$. $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is a facet of $\mathrm{P} \cap \mathrm{H}^{\ominus}_{d,\delta}$, i.e. $\dim(\mathrm{P} \cap \mathrm{H}_{d,\delta}) = \dim(\mathrm{P}) - 1 \geqslant 2$. By Corollary 3.1 $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is the convex hull of its facets. However, each facet of $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is a subset of at least one of the sets $\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta}$. Thus, $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ can be omitted in (14) and we therefore have (13).

(b) Let us suppose that $\dim(\mathrm{P} \cap \mathrm{H}_{d,\delta}) = \dim(\mathrm{P}) = 2$ and that $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is bounded. $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is the convex hull of its vertices. However, each of these vertices is contained in at least one of the sets $\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta}$. Thus, $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ can be omitted in (14) and we therefore have (13).

(c) Let us suppose that $\dim(\mathrm{P} \cap \mathrm{H}_{d,\delta}) = \dim(\mathrm{P}) = 2$ and that $\mathrm{P} \cap \mathrm{H}_{d,\delta}$ is unbounded. We then have $\mathrm{P} \cap \mathrm{H}_{d,\delta} = \{\hat{x} + \lambda r \mid \lambda \in \mathbf{R}^+_0\}$, where $\hat{x}$ is a vertex of $P \cap H^{\ominus}_{d,\delta}$, i.e. $\hat{x}$ is contained in at least one of the sets $F_j \cap H^{\ominus}_{d,\delta}$. Thus (14) is equivalent to (13). □

THEOREM 3.2. *Let* $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ *be a pointed polyhedron with* $\dim(P) = n$, *let* $S \neq \emptyset$ *be an N-set of* $V^{ps}(P_{(A,b)})$, *and let* $d^T x \geqslant \delta$ *be a cutting plane with* $d \in \mathbf{R}^n \setminus \{0\}$, $\delta \in \mathbf{R}$ *such that* $C_S(\tilde{x}_i) \subseteq H^{\oplus}_{d,\delta}$ *for all* $\tilde{x}_i \in S \cap H^{\oplus}_{d,\delta}$.

Let $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t \in \mathbf{R}^n$ *with* $\|\tilde{r}_k\| = 1$ *be all vectors fulfilling the following conditions. For* $\tilde{r}_k$ *there exists a pseudovertex* $\tilde{x}_{i_k} \in S \cap H^{-}_{d,\delta}$ *and a face* L_{i_k} *of* $C(\tilde{x}_{i_k})$ *with* $\dim(L_{i_k}) = 2$ *such that for*

$$Q_k = \bigcap_{\tilde{x}_i \in S \cap \mathrm{aff}(L_{i_k})} C(\tilde{x}_i)|_{\mathrm{aff}(L_{i_k})}$$

the following hold: $Q_k \cap H^{\ominus}_{d,\delta} \neq Q_k$, $\dim(Q_k \cap H^{\ominus}_{d,\delta}) = 2$, *and* $Q_k \cap H_{d,\delta}$ *is a half-line with direction* $\tilde{r}_k$. *With the above notation we have*

$$P \cap H^{\ominus}_{d,\delta} \subseteq \mathrm{conv}\left(\bigcup_{\tilde{x}_i \in S^{\ominus}} \left[C_S(\tilde{x}_i) \cap H^{\ominus}_{d,\delta} \right], S^N \right) + \mathrm{cone}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t)$$

where by $S^{\ominus}$ *and* S^N *we denote the sets* $S \cap H^{\ominus}_{d,\delta}$ *and* $\{\tilde{x}_l \in S \setminus H^{\ominus}_{d,\delta} \mid \exists\, \tilde{x}_i \in S \cap H^{\ominus}_{d,\delta} :$ $\tilde{x}_i$ *and* $\tilde{x}_l$ *are neighbors*$\}$, *respectively.*

Proof. As in the proof of Theorem 3.1 we consider the facets $F_1, F_2, \ldots, F_h$ of the P-containing polyhedron P_S (cf. Corollary 2.1). It follows from the definition of P_S that P_S fulfills the conditions of Corollary 3.2. Thus, to prove Theorem 3.2 it suffices to verify that

$$\begin{aligned} &\mathrm{conv}\left(\bigcup_{j=1}^{h} \left[F_j \cap H^{\ominus}_{d,\delta} \right] \right) + \mathrm{cone}(r) \\ &\quad \subseteq \mathrm{conv}\left(\bigcup_{\tilde{x}_i \in S^{\ominus}} \left[C_S(\tilde{x}_i) \cap H^{\ominus}_{d,\delta} \right], S^N \right) + \mathrm{cone}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t). \end{aligned} \tag{15}$$

By defining the N-set $S_j := S \cap \mathrm{aff}(F_j)$ we have $S_j \neq \emptyset$, and in the case of $F_j \cap H^{\ominus}_{d,\delta} \neq \emptyset$ we also have $S_j \cap H^{\ominus}_{d,\delta} \neq \emptyset$. The former follows from the definition of P_S, and the latter can be seen as follows.

Let F_{j_1} be an arbitrary facet of P_S such that $F_{j_1} \cap H^{\ominus}_{d,\delta} \neq \emptyset$, and let us assume $S_{j_1} \cap H^{\ominus}_{d,\delta} = \emptyset$, i.e. $S_{j_1} \subseteq H^{+}_{d,\delta}$. It follows from the condition $C_S(\tilde{x}_i) \subseteq H^{\oplus}_{d,\delta}$ for all $\tilde{x}_i \in S \cap H^{\oplus}_{d,\delta}$ and from $C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})} \subseteq C_S(\tilde{x}_i)$ that we have $C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})} \subseteq H^{+}_{d,\delta}$ for all $\tilde{x}_i \in S_{j_1} \subseteq H^{+}_{d,\delta}$. However, because of (12) this implies $F_{j_1} \subseteq H^{+}_{d,\delta}$, which contradicts $F_{j_1} \cap H^{\ominus}_{d,\delta} \neq \emptyset$.

Based on these considerations, we verify inclusion (15) by induction in n.

<u>n = 2:</u> Suppose that $P = \{x \in \mathbf{R}^2 \mid Ax \leqslant b\}$ is a pointed polyhedron with $\dim(P) = 2$. Let $S \neq \emptyset$ be an N-set of $V^{ps}(P_{(A,b)})$, and suppose that $d^T x \geqslant \delta$ is a cutting plane fulfilling the conditions of Theorem 3.2.

Let F_{j_1} be an arbitrary facet of the P-containing polyhedron P_S such that $F_{j_1} \cap H^{\ominus}_{d,\delta} \neq \emptyset$, and let $\tilde{x}_i \in S_{j_1} \cap H^{\ominus}_{d,\delta}$. According to the proof of Theorem 3.1, the facet F_{j_1} is contained in the edge $E_{i,2} = \{\tilde{x}_i + \lambda \tilde{u}_{i,2} \mid \lambda \in \mathbf{R}^+_0\}$ of the cone $C(\tilde{x}_i) = \tilde{x}_i + \text{cone}(\tilde{u}_{i,1}, \tilde{u}_{i,2})$ (cf. (6)). However $E_{i,2}$ is not necessarily an edge of the cone $C_S(\tilde{x}_i)$.

Case 1: Suppose that $E_{i,2}$ is an edge of the cone $C_S(\tilde{x}_i)$. Then we have $F_{j_1} \cap H^{\ominus}_{d,\delta} \subseteq E_{i,2} \cap H^{\ominus}_{d,\delta}$, which implies

$$F_{j_1} \cap H^{\ominus}_{d,\delta} \subseteq C_S(\tilde{x}_i) \cap H^{\ominus}_{d,\delta} \text{ with } \tilde{x}_i \in S^{\ominus}. \tag{16}$$

Case 2: Suppose that $E_{i,2}$ is not an edge of $C_S(\tilde{x}_i)$. According to the proof of Theorem 3.1 there exists an N_1-neighbor $\tilde{x}_l$ of $\tilde{x}_i$ with $\tilde{x}_l \in S_{j_1}$ such that $F_{j_1} \subseteq \text{conv}(\tilde{x}_i, \tilde{x}_l)$ (cf. (8)). If $\tilde{x}_l \in S_{j_1} \cap H^{\ominus}_{d,\delta} \subseteq S^{\ominus}$ we therefore have

$$F_{j_1} \cap H^{\ominus}_{d,\delta} \subseteq \text{conv}\Big(C_S(\tilde{x}_i) \cap H^{\ominus}_{d,\delta}, C_S(\tilde{x}_l) \cap H^{\ominus}_{d,\delta}\Big) \text{ with } \tilde{x}_i, \tilde{x}_l \in S^{\ominus} \tag{17}$$

and if $\tilde{x}_l \notin S_{j_1} \cap H^{\ominus}_{d,\delta}$ we have

$$F_{j_1} \cap H^{\ominus}_{d,\delta} \subseteq \text{conv}\Big(C_S(\tilde{x}_i) \cap H^{\ominus}_{d,\delta}, \tilde{x}_l\Big) \text{ with } \tilde{x}_i \in S^{\ominus}, \tilde{x}_l \in S^N. \tag{18}$$

Let $F_1, F_2, \ldots, F_h$ be the facets of P_S. Since F_{j_1} is an arbitrary facet of P_S with $F_{j_1} \cap H^{\ominus}_{d,\delta} \neq \emptyset$, it follows from (16), (17) and (18) that

$$\text{conv}\Bigg(\bigcup_{j=1}^{h} \Big[F_j \cap H^{\ominus}_{d,\delta}\Big]\Bigg) \subseteq \text{conv}\Bigg(\bigcup_{\tilde{x}_i \in S^{\ominus}} \Big[C_S(\tilde{x}_i) \cap H^{\ominus}_{d,\delta}\Big], S^N\Bigg). \tag{19}$$

To prove inclusion (15) for $n = 2$ it remains to be verified that $\text{cone}(r) \subseteq \text{cone}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t)$. For $r = 0$ this is obviously true. Therefore, let us suppose that $r \neq 0$. According to Corollary 3.2, the vector r is the direction of the half-line $P_S \cap H_{d,\delta}$ if $\emptyset \neq P_S \cap H^{\ominus}_{d,\delta} \neq P_S$, $\dim(P_S \cap H^{\ominus}_{d,\delta}) = 2$, and $P_S \cap H_{d,\delta}$ is unbounded. It holds that:

(1) There exists $\tilde{x}_{i_k} \in S \cap H^{-}_{d,\delta}$. Indeed, suppose that $S \cap H^{-}_{d,\delta} = \emptyset$, i.e. $S \subseteq H^{\oplus}_{d,\delta}$. Then it follows from the condition $C_S(\tilde{x}_i) \subseteq H^{\oplus}_{d,\delta}$ for all $\tilde{x}_i \in S \cap H^{\oplus}_{d,\delta}$ and by Theorem 3.1 that $P_S \subseteq \text{conv}(\bigcup_{\tilde{x}_i \in S} C_S(\tilde{x}_i)) \subseteq H^{\oplus}_{d,\delta}$, which contradicts $\dim(P_S \cap H^{\ominus}_{d,\delta}) = \dim(P_S) = 2$.

(2) Let $\tilde{x}_{i_k} \in S \cap H^{-}_{d,\delta}$. It follows from the definition of P_S that with respect to $L_{i_k} := C(\tilde{x}_{i_k})$ $\tilde{r} := r/\|r\|$ fulfills the conditions of the vectors $\tilde{r}_k$ in Theorem 3.2.

Hence, we have $\tilde{r} \in \{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t\}$ such that $\text{cone}(r) = \text{cone}(\tilde{r})$. This verifies inclusion (15). Based on the considerations at the beginning of the proof we have, therefore, verified Theorem 3.2 for $n = 2$.

$\underline{n-1 \to n}$: Let Theorem 3.2 hold for all full-dimensional and pointed polyhedra in $\mathbf{R}^k$ with $2 \leqslant k \leqslant n-1$. Suppose that $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ is a pointed polyhedron with $\dim(P) = n \geqslant 3$. Let S be an N-set of $V^{ps}(P_{(A,b)})$ and suppose that $d^T x \geqslant \delta$ is a cutting plane fulfilling the conditions of Theorem 3.2. Let F_{j_1} be an arbitrary facet of the P-containing polyhedron $P_S = \{x \in \mathbf{R}^n \mid A_S x \leqslant b_S\}$ with $F_{j_1} \cap H^{\ominus}_{d,\delta} \neq \emptyset$. P_S is pointed and because of $\dim(P_S) = n$ we have $\dim(F_j) = n-1$. Let $a^T_{j_1} x \leqslant \beta_{j_1}$ be the corresponding constraint of $A_S x \leqslant b_S$ such that $F_{j_1} = \{x \in P_S \mid a^T_{j_1} x = \beta_{j_1}\}$.

With the nonempty N-set $S_{j_1} := S \cap \mathrm{aff}(F_{j_1})$ of $V^{ps}(P_{S_{(A_S,b_S)}})$ we define the corresponding sets $S^{\ominus}_{j_1} := S_{j_1} \cap H^{\ominus}_{d,\delta}$ and $S^N_{j_1} := \{\tilde{x}_l \in S_{j_1} \setminus H^{\ominus}_{d,\delta} \mid \exists\, \tilde{x}_i \in S_{j_1} \cap H^{\ominus}_{d,\delta} :$ $\tilde{x}_i$ and $\tilde{x}_l$ are neighbors$\}$.

To apply the induction hypothesis we have to map $\mathrm{aff}(F_{j_1})$ into $\mathbf{R}^{n-1}$. To do this we consider the mapping $\phi_{j_1} : \mathrm{aff}(F_{j_1}) \mapsto \mathbf{R}^{n-1}$ with $\phi_{j_1}(x) = (V^T_{j_1} V_{j_1})^{-1} V^T_{j_1}(x - \tilde{x}_{j_1})$ and its inverse $\phi^{-1}_{j_1} : \mathbf{R}^{n-1} \mapsto \mathrm{aff}(F_{j_1})$ with $\phi^{-1}_{j_1}(y) = V_{j_1} y + \tilde{x}_{j_1}$, which we defined in the proof of Theorem 3.1 (cf. (9)). $\phi_{j_1}(F_{j_1})$ is a full-dimensional and pointed polyhedron in $\mathbf{R}^{n-1}$ (cf. (10)) and $\phi_{j_1}(S_{j_1})$ is an N-set of $\phi_{j_1}(F_{j_1})$. We have $\phi_{j_1}\big(H^{\ominus}_{d,\delta} \cap \mathrm{aff}(F_{j_1})\big) = \{y \in \mathbf{R}^{n-1} \mid d^T(V_{j_1} y + \tilde{x}_{j_1}) \leqslant \delta\}$ and by defining $\widehat{d} := d^T V_{j_1}$ $\widehat{\delta} := \delta - d^T \tilde{x}_{j_1}$ and $\widehat{H}^{\ominus}_{\hat{d},\hat{\delta}} := \{y \in \mathbf{R}^{n-1} \mid \widehat{d}^T y \leqslant \widehat{\delta}\}$ we therefore have $\phi_{j_1}\big(H^{\ominus}_{d,\delta} \cap \mathrm{aff}(F_{j_1})\big) = \widehat{H}^{\ominus}_{\hat{d},\hat{\delta}}$ and

$$\phi_{j_1}\big(F_{j_1} \cap H^{\ominus}_{d,\delta}\big) = \phi_{j_1}(F_{j_1}) \cap \widehat{H}^{\ominus}_{\hat{d},\hat{\delta}}. \tag{20}$$

The cutting plane $\widehat{d}^T y \geqslant \widehat{\delta}$ fulfills the conditions of Theorem 3.2. By defining $\tilde{y}_i := \phi_{j_1}(\tilde{x}_i)$ for $\tilde{x}_i \in S_{j_1}$ we get by the induction hypothesis

$$\begin{aligned} \phi_{j_1}(F_{j_1}) \cap \widehat{H}^{\ominus}_{\hat{d},\hat{\delta}} \subseteq & \mathrm{conv}\Bigg(\bigcup_{\tilde{y}_i \in \phi_{j_1}(S^{\ominus}_{j_1})} \Big[\widehat{C}_{\phi_{j_1}(S_{j_1})}(\tilde{y}_i) \cap \widehat{H}^{\ominus}_{\hat{d},\hat{\delta}}\Big], \phi_{j_1}(S^N_{j_1}) \Bigg) \\ & + \ \mathrm{cone}(\widehat{r}_{1_{j_1}}, \widehat{r}_{2_{j_1}}, \ldots, \widehat{r}_{t_{j_1}}), \end{aligned} \tag{21}$$

where $\widehat{C}_{\phi_{j_1}(S_{j_1})}(\phi_{j_1}(\tilde{x}_i))$ denotes the cone derived in $\mathbf{R}^{n-1}$ w.r.t. the polyhedron $\phi_{j_1}(F_{j_1})$ and the N-set $\phi_{j_1}(S_{j_1})$. It is not hard to verify

$$\phi^{-1}_{j_1}\Big(\widehat{C}_{\phi_{j_1}(S_{j_1})}(\phi_{j_1}(\tilde{x}_i)) \cap \widehat{H}^{\ominus}_{\hat{d},\hat{\delta}}\Big) = C_{S_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(F_{j_1})} \cap H^{\ominus}_{d,\delta}. \tag{22}$$

Therefore, by (21) and (20) we have

$$
\begin{aligned}
\mathrm{F}_{j_1} \cap \mathrm{H}^{\ominus}_{d,\delta} \subseteq \phi^{-1}_{j_1}\Bigg(&\mathrm{conv}\Bigg(\bigcup_{\tilde{y}_i \in \phi_{j_1}(\mathrm{S}^{\ominus}_{j_1})}\Big[\widehat{\mathrm{C}}_{\phi_{j_1}(\mathrm{S}_{j_1})}(\tilde{y}_i) \cap \widehat{\mathrm{H}}^{\ominus}_{\hat{d},\hat{\delta}}\Big], \phi_{j_1}(\mathrm{S}^N_{j_1})\Bigg) \\
&+ \ \mathrm{cone}(\widehat{r}_{1_{j_1}}, \widehat{r}_{2_{j_1}}, \dots, \widehat{r}_{t_{j_1}})\Bigg) \\
= \mathrm{conv}&\Bigg(\bigcup_{\tilde{y}_i \in \phi_{j_1}(\mathrm{S}^{\ominus}_{j_1})} \phi^{-1}_{j_1}\Big(\widehat{\mathrm{C}}_{\phi_{j_1}(\mathrm{S}_{j_1})}(\tilde{y}_i) \cap \widehat{\mathrm{H}}^{\ominus}_{\hat{d},\hat{\delta}}\Big), \mathrm{S}^N_{j_1}\Bigg) \\
&+ \mathrm{cone}(V_{j_1}\widehat{r}_{1_{j_1}}, V_{j_1}\widehat{r}_{2_{j_1}}, \dots, V_{j_1}\widehat{r}_{t_{j_1}}) \qquad \text{(cf. (9))} \\
\overset{(22)}{=} \mathrm{conv}&\Bigg(\bigcup_{\tilde{x}_i \in \mathrm{S}^{\ominus}_{j_1}}\Big[\mathrm{C}_{\mathrm{S}_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(\mathrm{F}_{j_1})} \cap \mathrm{H}^{\ominus}_{d,\delta}\Big], \mathrm{S}^N_{j_1}\Bigg) \\
&+ \mathrm{cone}(\tilde{r}_{1_{j_1}}, \tilde{r}_{2_{j_1}}, \dots, \tilde{r}_{t_{j_1}})
\end{aligned}
$$

with $\tilde{r}_{k_{j_1}} := V_{j_1}\widehat{r}_{k_{j_1}} / \| V_{j_1}\widehat{r}_{k_{j_1}} \|$. It is not hard to verify that $\tilde{r}_{k_{j_1}}$ fulfills the conditions in Theorem 3.2. Because of $\mathrm{S}^{\ominus}_{j_1} \subseteq \mathrm{S}^{\ominus}$, $\mathrm{S}^N_{j_1} \subseteq \mathrm{S}^N$, $\mathrm{C}_{\mathrm{S}_{j_1}}(\tilde{x}_i)|_{\mathrm{aff}(\mathrm{F}_{j_1})} \cap \mathrm{H}^{\ominus}_{d,\delta} \subseteq \mathrm{C}_{\mathrm{S}}(\tilde{x}_i) \cap \mathrm{H}^{\ominus}_{d,\delta}$ and $\{\tilde{r}_{1_{j_1}}, \tilde{r}_{2_{j_1}}, \dots, \tilde{r}_{t_{j_1}}\} \subseteq \{\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_t\}$ we have

$$
\mathrm{F}_{j_1} \cap \mathrm{H}^{\ominus}_{d,\delta} \subseteq \mathrm{conv}\Bigg(\bigcup_{\tilde{x}_i \in \mathrm{S}^{\ominus}}\Big[\mathrm{C}_{\mathrm{S}}(\tilde{x}_i) \cap \mathrm{H}^{\ominus}_{d,\delta}\Big], \mathrm{S}^N\Bigg) + \mathrm{cone}(\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_t).
$$

F_{j_1} is an arbitrary facet of P_{S} with $\mathrm{F}_{j_1} \cap \mathrm{H}^{\ominus}_{d,\delta} \neq \emptyset$. Let $\mathrm{F}_1, \mathrm{F}_2, \dots, \mathrm{F}_h$ be the facets of P_{S}. Thus we have

$$
\begin{aligned}
\mathrm{conv}\Bigg(\bigcup_{j=1}^{h}\Big[\mathrm{F}_j \cap \mathrm{H}^{\ominus}_{d,\delta}\Big]\Bigg) \subseteq &\mathrm{conv}\Bigg(\bigcup_{\tilde{x}_i \in \mathrm{S}^{\ominus}}\Big[\mathrm{C}_{\mathrm{S}}(\tilde{x}_i) \cap \mathrm{H}^{\ominus}_{d,\delta}\Big], \mathrm{S}^N\Bigg) \\
&+ \mathrm{cone}(\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_t).
\end{aligned} \tag{23}
$$

Since $\dim(\mathrm{P}_{\mathrm{S}}) = \dim(\mathrm{P}) \geqslant 3$, in (15) we have $r = 0$ (cf. Corollary 3.2). Hence, by (23) we have verified inclusion (15) for $n \geqslant 3$, which proves Theorem 3.1. □

4. Cutting Planes and Cone Decomposition

To derive a convexity cut as described in Section 1, we suppose to have a nondegenerate vertex x_0 of the polyhedron $\mathrm{P} = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ and a convex set K such that $x_0 \in \mathrm{int}(\mathrm{K})$ and $\mathrm{int}(\mathrm{K}) \cap (\mathrm{P} \cap \Omega) = \emptyset$. A convexity cut $c^T(x - x_0) \geqslant 1$ eliminates x_0 together with a portion of $\mathrm{C}(x_0) \cap \mathrm{int}(\mathrm{K})$, and eliminates no points in $\mathrm{P} \cap \Omega$. However, in general, the cone $\mathrm{C}(x_0)$ is a poor approximation of P.

To derive deeper (P, Ω)-cuts we utilize the concepts of the previous section to get a better approximation of P. The main idea is the following. As a nondegenerate vertex of P, x_0 is a nondegenerate pseudovertex. By choosing a suitable N-set $\mathrm{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$ such that $\tilde{x}_0 \in \mathrm{S}$ and $\mathrm{S} \subseteq \mathrm{int(K)}$, we replace the cone $\mathrm{C}(x_0)$ by the collection of cones $\mathrm{C_S}(\tilde{x}_1), \mathrm{C_S}(\tilde{x}_2), \ldots, \mathrm{C_S}(\tilde{x}_l)$. It follows from Theorem 3.1 that $\mathrm{P} \subseteq \mathrm{conv}(\bigcup_{\tilde{x}_i \in \mathrm{S}} \mathrm{C_S}(\tilde{x}_i))$. With respect to this approximation of P we derive a (P, Ω)-cut. The basis for deriving such a cutting plane is given in the following theorem, which can be proved by the inclusion provided by Theorem 3.2.

THEOREM 4.1. *Let* $\mathrm{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$ *be an* N*-set of* $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ *such that* $\mathrm{S} \subseteq \mathrm{int(K)}$*, let* $d^T x \geqslant \delta$ *be a cutting plane such that* $\mathrm{C_S}(\tilde{x}_i) \subseteq \mathrm{H}^{\oplus}_{d,\delta}$ *for all* $\tilde{x}_i \in \mathrm{S} \cap \mathrm{H}^{\oplus}_{d,\delta}$*, and let* $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t$ *be all respective vectors fulfilling the conditions of Theorem 3.2. If*

(A) $d^T \tilde{x}_i \neq \delta$ *for all* $\tilde{x}_i \in \mathrm{S}$*;*
(B) $\mathrm{C_S}(\tilde{x}_i) \cap \mathrm{H}^{-}_{d,\delta} \subseteq \mathrm{int(K)}$ *for all* $\tilde{x}_i \in \mathrm{S} \cap \mathrm{H}^{-}_{d,\delta}$*;*
(C) $x + \mathrm{cone}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t) \subseteq \mathrm{int(K)}$ *for all* $x \in \mathrm{int(K)}$*,*
then $d^T x \geqslant \delta$ *is a* (P, Ω)*-cut.*

The existence of a cutting plane $d^T x \geqslant \delta$ fulfilling the conditions of Theorem 4.1 is not ensured for an arbitrary N-set S with $\mathrm{S} \subseteq \mathrm{int(K)}$. Furthermore, under the assumption of existency the depth of the cutting plane depends on a reasonable choice of the N-set. In this section we are concerned with the construction of a suitable N-set S.

S will be derived in a series of steps. Starting with the N-set $\mathrm{S}_0 = \{\tilde{x}_1\}$ we gradually enlarge S_0 such that $\mathrm{S}_0 \subseteq \mathrm{S}_1 \subseteq \cdots \subseteq \mathrm{S}_q, \subseteq \mathrm{int(K)}$ where $\mathrm{S}_1, \mathrm{S}_2, \ldots, \mathrm{S}_q$ are N-sets of $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$. When deriving these N-sets we have to ensure that there always exists at least one $\tilde{x}_i \in \mathrm{S}_t$ such that $\dim(\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_i)) > \dim(\mathrm{C}_{\mathrm{S}_{t+1}}(\tilde{x}_i))$, because otherwise we have

$$\mathrm{P} \subseteq \mathrm{conv}\Big(\bigcup_{\tilde{x}_i \in \mathrm{S}_t} \mathrm{C}_{\mathrm{S}_t}(\tilde{x}_i)\Big) \subseteq \mathrm{conv}\Big(\bigcup_{\tilde{x}_i \in \mathrm{S}_{t+1}} \mathrm{C}_{\mathrm{S}_{t+1}}(\tilde{x}_i)\Big), \tag{24}$$

and there is no benefit in enlarging the N-set S_t to the N-set S_{t+1}, i.e. the approximation of P derived with respect to S_{t+1} is not better than the approximation derived with respect to S_t.

To construct such N-sets we extend the notion of neighborhood of pseudovertices to cone edges. This is based on the following observation. Let S be an N-set of $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$, and let $\tilde{x}_1, \tilde{x}_2 \in \mathrm{S}$ be neighbors. Thus the corresponding $(n, n+1)$-submatrices $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ of full rank of (A, b) differ in only one row, i.e. there exists an $(n-1, n+1)$-matrix $(\breve{A}, \breve{b})$ that is a submatrix of $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ (see (2)).

For an edge $\bar{\mathrm{E}}_1$ of the cone $\mathrm{C}(\tilde{x}_1) = \{x \in \mathbf{R}^n \mid \tilde{A}_1 x \leqslant \tilde{b}_1\}$ $n-1$ constraints of $\tilde{A}_1 x \leqslant \tilde{b}_1$ are binding. If for $\bar{\mathrm{E}}_1$ all $n-1$ constraints of $\breve{A}x \leqslant \breve{b}$ are binding, then $\bar{\mathrm{E}}_1$ or its negative extension contains $\tilde{x}_2$. Thus in this case $\bar{\mathrm{E}}_1$ is not an edge of $\mathrm{C_S}(\tilde{x}_1)$. Hence for every edge of $\mathrm{C_S}(\tilde{x}_1)$ $n-2$ constraints of $\breve{A}x \leqslant \breve{b}$ are binding. The same holds for the cone $\mathrm{C_S}(\tilde{x}_2)$.

DEFINITION 4.1. Let $\mathrm{P} = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ be a polyhedron with $\dim(\mathrm{P}) = n$, and let S be an N-set of $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$.

1. Let $\tilde{x}_1, \tilde{x}_2 \in \mathrm{S}$ be neighbors, and let $(\breve{A}, \breve{b})$ be the corresponding $(n-1, n+1)$-submatrix of $(\tilde{A}_1, \tilde{b}_1)$ and $(\tilde{A}_2, \tilde{b}_2)$ (see (2)). An edge $\bar{\mathrm{E}}_1$ of $\mathrm{C}_\mathrm{S}(\tilde{x}_1)$ and an edge $\bar{\mathrm{E}}_2$ of $\mathrm{C}_\mathrm{S}(\tilde{x}_2)$ are called neighbors if for $\bar{\mathrm{E}}_1$ and $\bar{\mathrm{E}}_2$ the same $n-2$ constraints of $\breve{A}x \leqslant \breve{b}$ are binding.
2. Let $\mathrm{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$, and let $\mathrm{R}_\mathrm{S} = \{\bar{\mathrm{E}}_1, \bar{\mathrm{E}}_2, \ldots, \bar{\mathrm{E}}_l\}$ be a set of cone edges, where $\bar{\mathrm{E}}_i$ is an edge of $\mathrm{C}_\mathrm{S}(\tilde{x}_i)$. The set of cone edges R_S is N-isomorph if for every pair $\tilde{x}_{i_1}, \tilde{x}_{i_2} \in \mathrm{S}$ of neighbors the corresponding edges $\bar{\mathrm{E}}_{i_1}, \bar{\mathrm{E}}_{i_2} \in \mathrm{R}_\mathrm{S}$ are also neighbors.

EXAMPLE 4.1. The pseudovertices $\tilde{x}_1$ and $\tilde{x}_2$ of Examples 2.1 and 2.2 are N_3-neighbors. Thus $\mathrm{S} = \{\tilde{x}_1, \tilde{x}_2\}$ is an N-set. With $\breve{A}^T = (a_4, a_5)$ and $\breve{b}^T = (\beta_4, \beta_5)$ we can see that for the cones $\mathrm{C}_\mathrm{S}(\tilde{x}_1)$ and $\mathrm{C}_\mathrm{S}(\tilde{x}_2)$ the edges $\bar{\mathrm{E}}_1 := \mathrm{E}_{1,1}$, $\bar{\mathrm{E}}_2 := \mathrm{E}_{2,1}$ and the edges $\bar{\mathrm{E}}_1 := \mathrm{E}_{1,2}$, $\bar{\mathrm{E}}_2 := \mathrm{E}_{2,2}$ are neighbors, respectively (see Figure 3(b)). Hence, we have the N-isomorph sets $\mathrm{R}_\mathrm{S}^1 = \{\mathrm{E}_{1,1}, \mathrm{E}_{2,1}\}$ and $\mathrm{R}_\mathrm{S}^2 = \{\mathrm{E}_{1,2}, \mathrm{E}_{2,2}\}$.

With the following theorem we lay the foundation for the construction of suitable N-sets of $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$.

THEOREM 4.2. *Let* $\mathrm{P} = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ *be a polyhedron with* $\dim(\mathrm{P}) = n$, *let* $\mathrm{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$ *be an N-set of* $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$, *and let the set of cone edges* $\mathrm{R}_\mathrm{S} = \{\bar{\mathrm{E}}_1, \bar{\mathrm{E}}_2, \ldots, \bar{\mathrm{E}}_l\}$ *be N-isomorph.*

Furthermore, let $a_{j^*}^T x \leqslant \beta_{j^*}$ and $a_{l^*}^T x \leqslant \beta_{l^*}$ be constraints of $Ax \leqslant b$ such that for $i, k = 1, 2, \ldots, l$ the following hold:

(A) $a_{j^*}^T \tilde{x}_i = \beta_{j^*}$ *and* $a_{l^*}^T \tilde{x}_i \neq \beta_{l^*}$;

(B) $\bar{\mathrm{E}}_i \subseteq \{x \in \mathbf{R}^n \mid a_{j^*}^T x \leqslant \beta_{j^*}\}$ *and* $\bar{\mathrm{E}}_i \not\subseteq \{x \in \mathbf{R}^n \mid a_{j^*}^T x = \beta_{j^*}\}$;

(C) The hyperplane $a_{l^*}^T x = \beta_{l^*}$ intersects $\bar{\mathrm{E}}_i \cup \bar{\mathrm{E}}_i^-$ at a point $\tilde{x}_{l+i}$, where if it intersects $\bar{\mathrm{E}}_i$, then $a_{l^*}^T \tilde{x}_i < \beta_{l^*}$, and $a_{l^*}^T \tilde{x}_i > \beta_{l^*}$ otherwise;

(D) for $\tilde{x}_{l+i}$ exactly n constraints of $Ax \leqslant b$ are binding;

(E) $\tilde{x}_{l+i} \neq \tilde{x}_{l+k}$ for $i \neq k$.

Let $\mathrm{S}' := \{\tilde{x}_{l+1}, \tilde{x}_{l+2}, \ldots, \tilde{x}_{2l}\}$. *Then* $\widehat{\mathrm{S}} = \mathrm{S} \cup \mathrm{S}'$ *is an N-set of* $\mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ *and we have*

$$\dim(\mathrm{C}_{\hat{\mathrm{S}}}(\tilde{x}_i)) = \dim(\mathrm{C}_{\hat{\mathrm{S}}}(\tilde{x}_{l+i})) = \dim(\mathrm{C}_\mathrm{S}(\tilde{x}_i)) - 1$$

for all $\tilde{x}_i \in \mathrm{S}$, $\tilde{x}_{l+i} \in \mathrm{S}'$.

Proof. Since $\tilde{x}_i \in \mathrm{S}$ is a nondegenerate pseudovertex there exists a unique $(n, n+1)$-submatrix $(\tilde{A}_i, \tilde{b}_i)$ of full rank of (A, b) such that $\tilde{A}_i \tilde{x}_i = \tilde{b}_i$. Because of condition (A) for all $\tilde{x}_i \in \mathrm{S}$ w.l.o.g. we have

$$\tilde{A}_i = \begin{pmatrix} a_{j^*}^T \\ \tilde{A}_{i \setminus \{1\}} \end{pmatrix} \quad \text{and} \quad \tilde{b}_i = \begin{pmatrix} \beta_{j^*} \\ \tilde{b}_{i \setminus \{1\}} \end{pmatrix}, \tag{25}$$

where by $(\tilde{A}_{i\setminus\{\cdot\}}, \tilde{b}_{i\setminus\{\cdot\}})$ we denote the matrix we obtain by eliminating the jth row of $(\tilde{A}_i, \tilde{b}_i)$ for $j \in \{\cdot\}$. For an edge of the cone $\mathrm{C}(\tilde{x}_i) = \{x \in \mathbf{R}^n \mid \tilde{A}_i x \leqslant \tilde{b}_i\}$ $n-1$ constraints of $\tilde{A}_i x \leqslant \tilde{b}_i$ are binding. Thus, because of (25) and condition (B) for $\bar{\mathrm{E}}_i \in \mathrm{R_S}$ we have

$$\bar{\mathrm{E}}_i = \{x \in \mathbf{R}^n \mid a_{j^*}^T x \leqslant \beta_{j^*},\ \tilde{A}_{i\setminus\{1\}} x = \tilde{b}_{i\setminus\{1\}}\}.$$

According to condition (C) the hyperplane $a_{l^*}^T x = \beta_{l^*}$ intersects the line $\bar{\mathrm{E}}_i \cup \bar{\mathrm{E}}_i^-$ at $\tilde{x}_{l+i}$. With

$$\tilde{A}_{l+i} = \begin{pmatrix} a_{l^*}^T \\ \tilde{A}_{i\setminus\{1\}} \end{pmatrix} \quad \text{and} \quad \tilde{b}_{l+i} = \begin{pmatrix} \beta_{l^*} \\ \tilde{b}_{i\setminus\{1\}} \end{pmatrix} \tag{26}$$

we therefore have $\tilde{A}_{l+i}\tilde{x}_{l+i} = \tilde{b}_{l+i}$, where $(\tilde{A}_{l+i}, \tilde{b}_{l+i})$ is an $(n, n+1)$-submatrix of full rank of (A, b), i.e. $\tilde{x}_{l+i} \in \mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$. Because of condition (D) the pseudovertex $\tilde{x}_{l+i}$ is nondegenerate.

It follows from (25) and (26) that $\tilde{x}_i$ and $\tilde{x}_{l+i}$ are neighbors. Because of conditions (B) and (C) $\tilde{x}_i$ and $\tilde{x}_{l+i}$ are N_1- or N_3-neighbors. Furthermore, since $(\tilde{A}_{i\setminus\{1\}}, \tilde{b}_{i\setminus\{1\}})$ in (25) and (26) is uniquely determined, $\tilde{x}_{l+i}$ is the only pseudovertex in $\mathrm{S}' = \{\tilde{x}_{l+1}, \tilde{x}_{l+2}, \ldots, \tilde{x}_{2l}\}$ that is a neighbor of $\tilde{x}_i$, and conversely, $\tilde{x}_i$ is the only pseudovertex in $\mathrm{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\}$ that is a neighbor of $\tilde{x}_{l+i}$.

S is an N-set and therefore contains no pair of N_2-neighbors. Furthermore, $\tilde{x}_i$ and $\tilde{x}_{l+i}$ are N_1- or N_3-neighbors. To prove that $\widehat{\mathrm{S}} = \mathrm{S} \cup \mathrm{S}'$ is an N-set it remains to be verified that S' contains no N_2-neighbors.

By (25) and (26) we can see that $\tilde{x}_{l+i}, \tilde{x}_{l+j} \in \mathrm{S}'$ are neighbors iff $\tilde{x}_i, \tilde{x}_j \in \mathrm{S}$ are neighbors, i.e. $(\tilde{A}_{i\setminus\{1\}}, \tilde{b}_{i\setminus\{1\}})$ and $(\tilde{A}_{j\setminus\{1\}}, \tilde{b}_{j\setminus\{1\}})$ differ in only one row. Suppose that $\tilde{x}_i, \tilde{x}_j \in \mathrm{S}$ are neighbors and that $(\tilde{A}_{i\setminus\{1\}}, \tilde{b}_{i\setminus\{1\}})$ and $(\tilde{A}_{j\setminus\{1\}}, \tilde{b}_{j\setminus\{1\}})$ differ in the last row, i.e. $(\tilde{a}_{i,n}^T, \tilde{\beta}_{i,n}) \neq (\tilde{a}_{j,n}^T, \tilde{\beta}_{j,n})$. We have to verify that $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are N_1- or N_3-neighbors. By defining

$$\mathrm{A} := \{x \in \mathbf{R}^n \mid \tilde{A}_{i\setminus\{1,n\}} x = \tilde{b}_{i\setminus\{1,n\}}\}$$

with $\dim(\mathrm{A}) = 2$ for the neighbors $\tilde{x}_i, \tilde{x}_j$ the corresponding line (3) can be described by $\mathrm{G}_{j^*} = \{x \in \mathrm{A} \mid a_{j^*}^T x = \beta_{j^*}\}$, and for the neighbors $\tilde{x}_{l+i}, \tilde{x}_{l+j}$ the line (3) can be described by $\mathrm{G}_{l^*} = \{x \in \mathrm{A} \mid a_{l^*}^T x = \beta_{l^*}\}$.
The pseudovertices $\tilde{x}_i$ and $\tilde{x}_j$ are defined by the intersection of the line G_{j^*} with the hyperplanes $\tilde{a}_{i,n}^T x = \tilde{\beta}_{i,n}$ and $\tilde{a}_{j,n}^T x = \tilde{\beta}_{j,n}$, respectively, and $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are defined by the intersection of G_{l^*} with the hyperplanes $\tilde{a}_{i,n}^T x = \tilde{\beta}_{i,n}$ and $\tilde{a}_{j,n}^T x = \tilde{\beta}_{j,n}$, respectively. We have to distinguish between N_1- and N_3-neighborhoods of $\tilde{x}_i$ and $\tilde{x}_j$.

Case 1: Suppose that $\tilde{x}_i$ and $\tilde{x}_j$ are N_1-neighbors, i.e. $\tilde{a}_{i,n}^T \tilde{x}_j < \tilde{\beta}_{i,n}$ and $\tilde{a}_{j,n}^T \tilde{x}_i < \tilde{\beta}_{j,n}$. If $\mathrm{conv}(\tilde{x}_i, \tilde{x}_{l+i}) \cap \mathrm{conv}(\tilde{x}_j, \tilde{x}_{l+j}) = \emptyset$, then $\tilde{a}_{i,n}^T \tilde{x}_{l+j} < \tilde{\beta}_{i,n}$ and $\tilde{a}_{j,n}^T \tilde{x}_{l+i} < \tilde{\beta}_{j,n}$, i.e. $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are also N_1-neighbors (see Figure 5(a)). If $\mathrm{conv}(\tilde{x}_i, \tilde{x}_{l+i}) \cap$

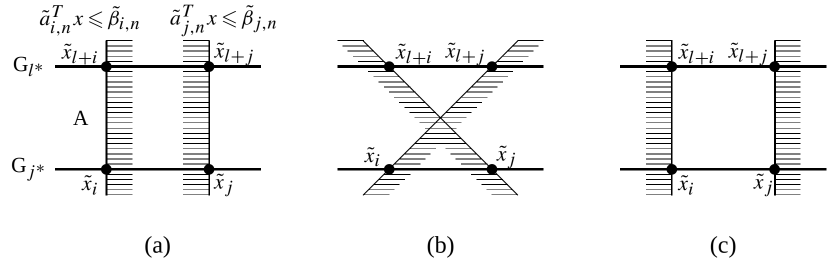

Figure 5. Neighborhood relations when enlarging an N-set S to an N-set $\widehat{S}$.

$\text{conv}(\tilde{x}_j, \tilde{x}_{l+j}) \neq \emptyset$, then $\tilde{a}_{i,n}^T \tilde{x}_{l+j} > \tilde{\beta}_{i,n}$ and $\tilde{a}_{j,n}^T \tilde{x}_{l+i} > \tilde{\beta}_{j,n}$, i.e. $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are N_3-neighbors (see Figure 5(b)).

Case 2: Suppose that $\tilde{x}_i$ and $\tilde{x}_j$ are N_3-neighbors, i.e. $\tilde{a}_{j,n}^T \tilde{x}_i > \tilde{\beta}_{j,n}$ and $\tilde{a}_{i,n}^T \tilde{x}_j > \tilde{\beta}_{i,n}$. In analogy to Case 1, we can verify that if $\text{conv}(\tilde{x}_i, \tilde{x}_{l+i}) \cap \text{conv}(\tilde{x}_j, \tilde{x}_{l+j}) = \emptyset$, then $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are also N_3-neighbors (see Figure 5(c)) and that if $\text{conv}(\tilde{x}_i, \tilde{x}_{l+i}) \cap \text{conv}(\tilde{x}_j, \tilde{x}_{l+j}) \neq \emptyset$, then $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are N_1-neighbors.

Thus if $\tilde{x}_i$ and $\tilde{x}_j$ are N_1- or N_3-neighbors, then $\tilde{x}_{l+i}$ and $\tilde{x}_{l+j}$ are also N_1- or N_3-neighbors, i.e. S' contains no N_2-neighbors. Hence $\widehat{S} = S \cup S'$ is an N-set of $V^{ps}(P_{(A,b)})$. Since $\tilde{x}_i \in S$ is a neighbor of only one pseudovertex in $S' = \widehat{S} \setminus S$, we have $\dim(C_{\hat{S}}(\tilde{x}_i)) = \dim(C_S(\tilde{x}_i)) - 1$. The pseudovertices $\tilde{x}_{l+i}, \tilde{x}_{l+j} \in S'$ are neighbors, iff the pseudovertices $\tilde{x}_i, \tilde{x}_j \in S$ are neighbors. Furthermore, $\tilde{x}_i$ is the only neighbor of $\tilde{x}_{l+i} \in S'$ contained in S. Hence we have $\dim(C_{\hat{S}}(\tilde{x}_{l+i})) = \dim(C_{\hat{S}}(\tilde{x}_i))$. □

When enlarging an N-set S to an N-set $\widehat{S}$ according to Theorem 4.1 we replace the cone $C_S(\tilde{x}_i)$ with $\dim(C_S(\tilde{x}_i)) = k_i$ by $(k_i - 1)$-dimensional cones $C_{\hat{S}}(\tilde{x}_i)$ and $C_{\hat{S}}(\tilde{x}_{l+i})$. This is referred to as *cone decomposition*.

Let x_0 be a nondegenerate vertex of $P = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ that is to be eliminated, and let K be a convex set such that $x_0 \in \text{int}(K)$ and $\text{int}(K) \cap (P \cap \Omega) = \emptyset$. $S_0 := \{x_0\}$ is an N-set of $V^{ps}(P_{(A,b)})$, and we have $C_{S_0}(x_0) = C(x_0)$, where $C(x_0)$ is identical with the cone with respect to which we derive a convexity cut $c^T(x - x_0) \geqslant 1$.

To derive deeper (P, Ω)-cuts, by repeatedly applying Theorem 4.1 we decompose the cone $C(x_0)$ gradually into cones with smaller dimension that are also vertexed in int(K). This is done by the following procedure, where *depth* is a prechosen maximal decomposition depth, $\bar{E}_i = \{y_{i,j_i}(\lambda) = \tilde{x}_i + \lambda \tilde{u}_{i,j_i} \mid \lambda \in \mathbf{R}_0^+\}$ an edge of $C_{S_t}(\tilde{x}_i)$, and $\bar{E}_i^-$ its negative extension, i.e. $\bar{E}_i^- = \{y_{i,j_i}(\lambda) = \tilde{x}_i + \lambda \tilde{u}_{i,j_i} \mid \lambda \in \mathbf{R}_0^-\}$.

Cone Decomposition Procedure (CDP)

set $S_0 := \{\tilde{x}_1\}$ with $\tilde{x}_1 := x_0$;

set $deco := true$ and $t := 0$;

While ($deco$ and $t < depth$) do

if there exists an N-isomorph set of cone edges $\mathrm{R}_{\mathrm{S}_t} = \{\bar{\mathrm{E}}_1, \bar{\mathrm{E}}_2, \dots, \bar{E}_{2^t}\}$ and a constraint $a_{l_t^*}^T x \leqslant \beta_{l_t^*}$ of $Ax \leqslant b$ such that for $i, k = 1, 2, \dots, 2^t$, the following conditions hold:

1. $a_{l_t^*}^T x = \beta_{l_t^*}$ intersects $\bar{\mathrm{E}}_i \cup \bar{\mathrm{E}}_i^-$ at a point $\tilde{x}_{2^t+i} \in \mathrm{int(K)}$;
2. if $a_{l_t^*}^T x = \beta_{l_t^*}$ intersects $\bar{\mathrm{E}}_i$, then $a_{l_t^*}^T \tilde{x}_i < \beta_{l_t^*}$, and $a_{l_t^*}^T \tilde{x}_i > \beta_{l_t^*}$ otherwise;
3. for $\tilde{x}_{2^t+i}$ exactly n constraints of $Ax \leqslant b$ are binding;
4. $\tilde{x}_{2^t+i} \neq \tilde{x}_{2^t+k}$ for $i \neq k$;

then set $\mathrm{S}_{t+1} := \mathrm{S}_t \cup \{\tilde{x}_{2^t+1}, \tilde{x}_{2^t+2}, \dots, \tilde{x}_{2^{t+1}}\}$ and $t := t + 1$;

else set $deco := false$;

set $\mathrm{S} := \mathrm{S}_t$.

For the sets S_t derived by CDP the following lemma holds, which can be proved by induction in t.

LEMMA 4.1. *Let* $\mathrm{S}_0 = \{\tilde{x}_1\} \subseteq \mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ *be the initial* N*-set of CDP, and let* $\tilde{A}_{1_0} := \tilde{A}_1$ *and* $\tilde{b}_{1_0} := \tilde{b}_1$*, where* $(\tilde{A}_1, \tilde{b}_1)$ *is the corresponding* $(n, n+1)$*-submatrix of full rank of* (A, b) *such that* $\tilde{A}_1 \tilde{x}_1 = \tilde{b}_1$*. For* S_t $(t \geqslant 1)$ *there exists an* $(n-t, n+1)$*-submatrix* $(\tilde{A}_{1_t}, \tilde{b}_{1_t})$ *of* $(\tilde{A}_{1_{t-1}}, \tilde{b}_{1_{t-1}})$ *such that for all* $\tilde{x}_i \in \mathrm{S}_t$ *all constraints of* $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$ *are binding.*

Furthermore, for each N*-isomorph set* $\mathrm{R}_{\mathrm{S}_t}$ *there exists a unique constraint* $\tilde{a}_{1_t,j}^T x \leqslant \tilde{\beta}_{1_t,j}$ *of* $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$ *such that* $\bar{\mathrm{E}}_i \subseteq \{x \in \mathbf{R}^n \mid \tilde{a}_{1_t,j}^T x \leqslant \tilde{\beta}_{1_t,j}\}$ *and* $\bar{\mathrm{E}}_i \not\subseteq \{x \in \mathbf{R}^n \mid \tilde{a}_{1_t,j}^T x = \tilde{\beta}_{1_t,j}\}$ *for all* $\bar{\mathrm{E}}_i \in \mathrm{R}_{\mathrm{S}_t}$*, and conversely, for each constraint* $\tilde{a}_{1_t,j}^T x \leqslant \tilde{\beta}_{1_t,j}$ *of* $\tilde{A}_{1_t} x \leqslant \tilde{b}_{1_t}$ *there exists a unique* N*-isomorph set* $\mathrm{R}_{\mathrm{S}_t}$ *such that* $\bar{\mathrm{E}}_i \subseteq \{x \in \mathbf{R}^n \mid \tilde{a}_{1_t,j}^T x \leqslant \tilde{\beta}_{1_t,j}\}$ *and* $\bar{\mathrm{E}}_i \not\subseteq \{x \in \mathbf{R}^n \mid \tilde{a}_{1_t,j}^T x = \tilde{\beta}_{1_t,j}\}$ *for all* $\bar{\mathrm{E}}_i \in \mathrm{R}_{\mathrm{S}_t}$.

Therefore, by choosing an inequality $a_{l_t^*}^T x \leqslant \beta_{l_t^*}$ and an N-isomorph set $\mathrm{R}_{\mathrm{S}_t}$ fulfilling the conditions in CDP we also have implicitly determined a constraint $a_{j_t^*}^T x \leqslant \beta_{j_t^*}$ such that all the conditions of Theorem 4.1 are fulfilled. Hence, the resulting set S_{t+1} is also an N-set and we have $\dim(\mathrm{C}_{\mathrm{S}_{t+1}}(\tilde{x}_i)) = \dim(\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_j)) - 1$ for all $\tilde{x}_i \in \mathrm{S}_{t+1}$ and $\tilde{x}_j \in \mathrm{S}_t$.

Starting with $\mathrm{S}_0 = \{x_0\}$, by repeatedly applying Theorem 4.1 in a way which ensures that the resulting N-sets are contained in int(K), after t stages we have a sequence of N-sets S_t such that $\mathrm{S}_0 \subseteq \mathrm{S}_1 \subseteq \cdots \subseteq \mathrm{S}_t$ with $|\mathrm{S}_t| = 2^t$, and $\dim(\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_i)) = n - t$ for all $\tilde{x}_i \in \mathrm{S}_t$.

EXAMPLE 4.2. Given are a polyhedron P, a nondegenerate vertex x_0 of P, and a convex set K such that $x_0 \in \mathrm{int(K)}$. K has been omitted in Figure 6(a), but the

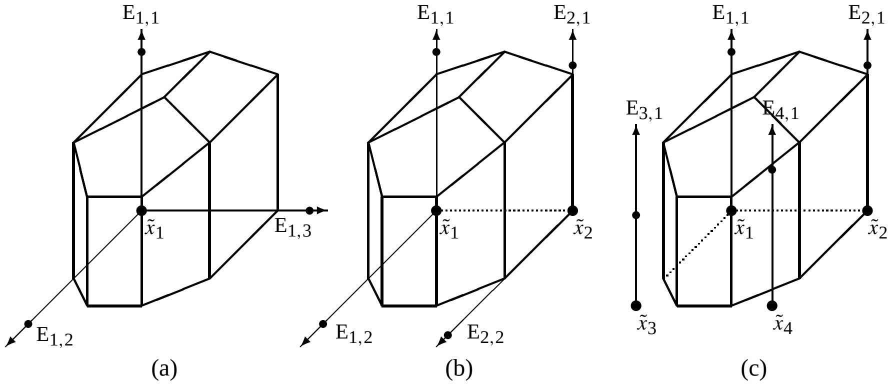

Figure 6. Decomposition of the cone $C(\tilde{x}_1)$ by CDP.

intersection points of the boundary of cl(K) and the edges of the respective cones are indicated by dots. In CDP we start with an N-set $S_0 = \{\tilde{x}_1\}$, where $\tilde{x}_1 := x_0$, and a cone $C_{S_0}(\tilde{x}_1) = \tilde{x}_1 + \text{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2}, \tilde{u}_{1,3})$ (see Figure 6(a)). There exist three N-isomorph sets $R^j_{S_0} = \{E_{1,j}\}$ $(j = 1, 2, 3)$. All these sets fulfill the if-conditions of CDP. We choose $R^3_{S_0}$ and the constraint which describes the right facet of P. By CDP we get an N-set $S_1 = \{\tilde{x}_1, \tilde{x}_2\}$ and the cones $C_{S_1}(\tilde{x}_1) = \tilde{x}_1 + \text{cone}(\tilde{u}_{1,1}, \tilde{u}_{1,2})$ and $C_{S_1}(\tilde{x}_2) = \tilde{x}_2 + \text{cone}(\tilde{u}_{2,1}, \tilde{u}_{2,2})$ (see Figure 6(b)). We have $P \subseteq \text{conv}(C_{S_1}(\tilde{x}_1), C_{S_1}(\tilde{x}_2))$. There exist two N-isomorph sets $R^j_{S_1} = \{E_{1,j}, E_{2,j}\}$ $(j = 1, 2)$. By choosing $R^2_{S_1}$ and the constraint describing the front facet of P we get $S_2 = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$ and $C_{S_2}(\tilde{x}_i) = \tilde{x}_i + \text{cone}(\tilde{u}_{i,1})$ with $i = 1, 2, 3, 4$ (see Figure 6.c). We have $P \subseteq \text{conv}(C_{S_2}(\tilde{x}_1), C_{S_2}(\tilde{x}_2), C_{S_2}(\tilde{x}_3), C_{S_2}(\tilde{x}_4))$. There exists only one N-isomorph set $R_{S_2} = \{E_{1,1}, E_{2,1}, E_{3,1}, E_{4,1}\}$. Since there exists no P-describing constraint which fulfills together with R_{S_2} the if-conditions of CDP, CDP stops with $S := S_2 \subseteq \text{int}(K)$.

5. Decomposition Cuts

When CDP stops, we have an N-set $S = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2^t}\}$ such that the respective cones $C_S(\tilde{x}_i)$ are $(n–t)$-dimensional and vertexed in int(K). In the case of $t = n$, we have $C_S(\tilde{x}_i) = \tilde{x}_i \in \text{int}(K)$, and by Theorem 3.1 we have $P \subseteq \text{conv}(S) \subseteq \text{int}(K)$. Consequently, we have $P \cap \Omega = \emptyset$, and we do not have to derive a (P, Ω)-cut. In the case of $t < n$, to derive a (P, Ω)-cut we specify in this section the conditions of Theorem 4.1 for N-sets obtained from CDP.

To prove that a cutting plane is a (P, Ω)-cut it suffices to verify that the cutting plane fulfills the condition of Theorem 4.1. To verify conditions (A) and (B) of Theorem 4.1 is generally not a problem. However, to verify condition (C) of Theorem 4.1 can be difficult. The following corollaries will be helpful in verifying condition (C).

COROLLARY 5.1. *Let the N-sets* $\mathrm{S}_t, \mathrm{S}_{t+1} \subseteq \mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ *be obtained from CDP, let* $\tilde{x}_{i_k} \in \mathrm{S}_t$*, and let* L_{i_k} *be a face of* $\mathrm{C}(\tilde{x}_{i_k})$ *with* $\dim(\mathrm{L}_{i_k}) = 2$*. If* $\tilde{x}_i \in \mathrm{S}_t$ *and* $\tilde{x}_i \notin \mathrm{aff}(\mathrm{L}_{i_k})$*, then we also have* $\tilde{x}_{2^t+i} \notin \mathrm{aff}(\mathrm{L}_{i_k})$*, where* $\tilde{x}_{2^t+i} \in \mathrm{S}_{t+1}$ *is derived in the* $(t+1)$*th stage of CDP.*

Proof. Since $\tilde{x}_{i_k} \in \mathrm{S}_t$ is a nondegenerate pseudovertex, there exists a unique $(n, n+1)$-submatrix $(\tilde{A}_{i_k}, \tilde{b}_{i_k})$ of full rank of (A, b) such that $\tilde{A}_{i_k}\tilde{x}_{i_k} = \tilde{b}_{i_k}$. Thus, we have $\mathrm{C}(\tilde{x}_{i_k}) = \{x \in \mathbf{R}^n \mid \tilde{A}_{i_k}x \leqslant \tilde{b}_{i_k}\}$. For the 2-dimensional face L_{i_k} of $\mathrm{C}(\tilde{x}_{i_k})$ $n-2$ constraints of $\tilde{A}_{i_k}x \leqslant \tilde{b}_{i_k}$ are binding. Let $(\tilde{A}'_{i_k}, \tilde{b}'_{i_k})$ be an $(n-2, n)$-submatrix of $(\tilde{A}_{i_k}, \tilde{b}_{i_k})$ such that $\mathrm{aff}(\mathrm{L}_{i_k}) = \{x \in \mathbf{R}^n \mid \tilde{A}'_{i_k}x = \tilde{b}'_{i_k}\}$. We prove Corollary 5.1 by contradiction. Suppose that $\tilde{x}_i \in \mathrm{S}_t$ and $\tilde{x}_i \notin \mathrm{aff}(\mathrm{L}_{i_k})$, and let us assume that $\tilde{x}_{2^t+i} \in \mathrm{aff}(\mathrm{L}_{i_k})$ for $\tilde{x}_{2^t+i} \in \mathrm{S}_{t+1}$. $\tilde{x}_{2^t+i}$ is the intersection point of the hyperplane $a^T_{l_t^*}x = \beta_{l_t^*}$ and the line $\bar{\mathrm{E}}_i \cup \bar{\mathrm{E}}_i^-$ (see CDP).

Let $(\tilde{A}_i, \tilde{b}_i)$ be the unique $(n, n+1)$-submatrix of (A, b) such that $\tilde{A}_i\tilde{x}_i = \tilde{b}_i$. For the line $\bar{\mathrm{E}}_i \cup \bar{\mathrm{E}}_i^-$ there exists a unique $(n-1, n+1)$-submatrix $(\tilde{A}_{i\setminus\{1\}}, \tilde{b}_{i\setminus\{1\}})$ of $(\tilde{A}_i, \tilde{b}_i)$ such that $\bar{\mathrm{E}}_i \cup \bar{\mathrm{E}}_i^- = \{x \in \mathbf{R}^n \mid \tilde{A}_{i\setminus\{1\}}x = \tilde{b}_{i\setminus\{1\}}\}$ (see (25)). Hence, we have $a^T_{l_t^*}\tilde{x}_{2^t+i} = \beta_{l_t^*}$, and $\tilde{A}_{i\setminus\{1\}}\tilde{x}_{2^t+i} = \tilde{b}_{i\setminus\{1\}}$. By assumption we have $\tilde{x}_{2^t+i} \in \mathrm{aff}(\mathrm{L}_{i_k})$ which implies $\tilde{A}'_{i_k}\tilde{x}_{2^t+i} = \tilde{b}'_{i_k}$. However, because of $\tilde{x}_i \notin \mathrm{aff}(\mathrm{L}_{i_k})$ there are at most $n-3$ constraints of $\tilde{A}'_{i_k}x = \tilde{b}'_{i_k}$ that are also constraints of $\tilde{A}_{i\setminus\{1\}}x = \tilde{b}_{i\setminus\{1\}}$. Therefore, since $\tilde{x}_{2^t+i}$ is nondegenerate, $a^T_{l_t^*}x = \beta_{l_t^*}$ has to be a constraint of $\tilde{A}'_{i_k}x = \tilde{b}'_{i_k}$. Since $\tilde{x}_{i_k} \in \mathrm{aff}(\mathrm{L}_{i_k})$ this implies that $a^T_{l_t^*}\tilde{x}_{i_k} = \beta_{l_t^*}$. But this contradicts Condition 2. in CDP. Hence, we have $\tilde{x}_{2^t+i} \notin \mathrm{aff}(\mathrm{L}_{i_k})$. □

COROLLARY 5.2. *Let the N-set* $\mathrm{S}_t \subseteq \mathrm{V}^{ps}(\mathrm{P}_{(A,b)})$ *be obtained from CDP, let* $\tilde{x}_{i_k} \in \mathrm{S}_t$*, and let* L_{i_k} *be a face of* $\mathrm{C}(\tilde{x}_{i_k})$ *with* $\dim(\mathrm{L}_{i_k}) = 2$*. For* $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k})$ *only the following cases can occur:*

1. $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}\}$*, where* $\tilde{x}_{i_1} = \tilde{x}_{i_k}$*;*
2. $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\}$*, where* $\tilde{x}_{i_1}$ *and* $\tilde{x}_{i_2}$ *are neighbors;*
3. $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}, \tilde{x}_{i_3}, \tilde{x}_{i_4}\}$*, where* $\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_{i_j})|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \tilde{x}_{i_j}$ *and* $\bigcap_{j=1}^4 \mathrm{C}(\tilde{x}_{i_j})|_{\mathrm{aff}(\mathrm{L}_{i_k})} \subseteq \mathrm{conv}(\bigcup_{j=1}^4 \tilde{x}_{i_j})$.

Proof. With the notation of the proof of Corollary 5.1 we have $\mathrm{aff}(\mathrm{L}_{i_k}) = \{x \in \mathbf{R}^n \mid \tilde{A}'_{i_k}x = \tilde{b}'_{i_k}\}$. Let $\tilde{x}_i \in \mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k})$ and let $(\tilde{A}_i, \tilde{b}_i)$ be the corresponding $(n, n+1)$-submatrix of full rank of (A, b) such that $\tilde{A}_i\tilde{x}_i = \tilde{b}_i$. Hence we have $\mathrm{C}(\tilde{x}_i) = \{x \in \mathbf{R}^n \mid \tilde{A}_ix \leqslant \tilde{b}_i\}$ with $\mathrm{C}(\tilde{x}_i) = \tilde{x}_i + \mathrm{cone}(\tilde{u}_{i,1}, \tilde{u}_{i,2}, \ldots, \tilde{u}_{i,n})$, where $\tilde{u}_{i,l}$ are directions of the edges of $\mathrm{C}(\tilde{x}_i)$. Since $\tilde{x}_i \in \mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k})$ is nondegenerate, $(\tilde{A}'_{i_k}, \tilde{b}'_{i_k})$ is a submatrix of $(\tilde{A}_i, \tilde{b}_i)$, and this verifies that $\dim(\mathrm{C}(\tilde{x}_i)|_{\mathrm{aff}(\mathrm{L}_{i_k})}) = 2$, i.e.

$$\mathrm{C}(\tilde{x}_i)|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \tilde{x}_i + \mathrm{cone}(\tilde{u}_{i,1}, \tilde{u}_{i,2}) \qquad \forall\, \tilde{x}_i \in \mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k}), \tag{27}$$

and

$$(\mathrm{E}_{i,l} \cup \mathrm{E}_{i,l}^{-}) \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \begin{cases} \mathrm{E}_{i,l} \cup \mathrm{E}_{i,l}^{-} & \text{for } l = 1, 2 \\ \{\tilde{x}_i\} & \text{otherwise} \end{cases}, \tag{28}$$

where $\mathrm{E}_{i,l} = \{\tilde{x}_i + \lambda \tilde{u}_{i,l} \mid \lambda \in \mathbf{R}_0^+\}$ and $\mathrm{E}_{i,l}^{-} = \{\tilde{x}_i + \lambda \tilde{u}_{i,l} \mid \lambda \in \mathbf{R}_0^-\}$.

We prove Corollary 5.2 by induction in t. For $t = 0$, Corollary 5.2 is obviously true. Suppose that it holds for all N-sets S_p with $p \leqslant t$. Let $\mathrm{S}_{t+1} = \mathrm{S}_t \cup \{\tilde{x}_{2^t+1}, \tilde{x}_{2^t+2}, \dots, \tilde{x}_{2^{t+1}}\}$ be obtained from $\mathrm{S}_t = \{\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_{2^t}\}$ by CDP, and let $\mathrm{R}_{\mathrm{S}_t} = \{\bar{\mathrm{E}}_1, \bar{\mathrm{E}}_2, \dots, \bar{\mathrm{E}}_{2^t}\}$ be the corresponding N-isomorph set of cone edges. We have to verify Corollary 5.2 for S_{t+1}. For this we distinguish three cases.

Case 1: Suppose that for $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k})$ case 1 of Corollary 5.2 holds, i.e. $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}\}$. Because of (28) there is no pseudovertex in $\mathrm{S}_t \setminus \{\tilde{x}_{i_1}\}$ lying on the edges $\mathrm{E}_{i_1,1}, \mathrm{E}_{i_1,2}$ of the cone $\mathrm{C}(\tilde{x}_i)|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ or on their negative extensions. Thus, $\mathrm{E}_{i_1,1}$ and $\mathrm{E}_{i_1,2}$ are also edges of $\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_{i_1})$.

Suppose that $\bar{\mathrm{E}}_{i_1} \neq \mathrm{E}_{i_1,l}$ for $l = 1, 2$. Then $\tilde{x}_{2^t+i_1} \notin \mathrm{aff}(\mathrm{L}_{i_k})$ and $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}\}$. The former follows from (28) and from the construction of CDP, and the latter follows from Corollary 5.1. Hence, for $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k})$ case 1 of Corollary 5.2 holds.

Suppose that $\bar{\mathrm{E}}_{i_1} = \mathrm{E}_{i_1,1}$ or $\bar{\mathrm{E}}_{i_1} = \mathrm{E}_{i_1,2}$. Hence, by (28) we have $\tilde{x}_{2^t+i_1} \in \mathrm{aff}(\mathrm{L}_{i_k})$, where by construction of CDP $\tilde{x}_{2^t+i_1}$ is a neighbor of $\tilde{x}_{i_1}$. It follows from Corollary 5.1 that $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\}$, where $\tilde{x}_{i_2} := \tilde{x}_{2^t+i_1}$, i.e. for $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k})$ Case 2 of Corollary 5.2 holds.

Case 2: Suppose that for $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k})$ case 2 of Corollary 5.2 holds, i.e. $\mathrm{S}_t \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\}$, where $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$ are neighbors. Since $\tilde{x}_{i_1}, \tilde{x}_{i_2}$ are nondegenerate neighbors, $\tilde{x}_{i_1}$ lies on an edge of $\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_{i_2})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ or on its negative extension, and $\tilde{x}_{i_2}$ lies on an edge of $\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ or on its negative extension. Hence, we have $\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \tilde{x}_{i_1} + \mathrm{cone}(\tilde{u}_{i_1,1})$ and $\mathrm{C}_{\mathrm{S}_t}(\tilde{x}_{i_2})|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \tilde{x}_{i_2} + \mathrm{cone}(\tilde{u}_{i_2,1})$. It is not hard to verify that the edges $\mathrm{E}_{i_1,1}$ and $\mathrm{E}_{i_2,1}$ are neighbors. Since $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$ are neighbors and $\mathrm{R}_{\mathrm{S}_t}$ is N-isomorph, for the cone edges $\bar{\mathrm{E}}_{i_1}, \bar{\mathrm{E}}_{i_2} \in \mathrm{R}_{\mathrm{S}_t}$ it holds that $\bar{\mathrm{E}}_{i_1} = \mathrm{E}_{i_1,1}$, iff $\bar{\mathrm{E}}_{i_2} = \mathrm{E}_{i_2,1}$. Hence, using arguments similar to those for Case 1 we can show that by construction of CDP and because of (28) and Corollary 5.1 the following hold.

(a) Suppose that $\bar{\mathrm{E}}_{i_1} \neq \mathrm{E}_{i_1,1}$. Then $\bar{\mathrm{E}}_{i_2} \neq \mathrm{E}_{i_2,1}$, i.e. the edges $\bar{\mathrm{E}}_{i_1}$ and $\bar{\mathrm{E}}_{i_2}$ correspond to edges $\mathrm{E}_{i_1,l}$ and $\mathrm{E}_{i_2,g}$ with $l, g \geqslant 3$. Hence, we have $\tilde{x}_{2^t+i_1}, \tilde{x}_{2^t+i_2} \notin \mathrm{aff}(\mathrm{L}_{i_k})$ and $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\}$, i.e. for $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k})$ case 2 of Corollary 5.2 holds.

(b) Suppose that $\bar{\mathrm{E}}_{i_1} = \mathrm{E}_{i_1,1}$. Then we have $\bar{\mathrm{E}}_{i_2} = \mathrm{E}_{i_2,1}$. Hence, we have $\tilde{x}_{2^t+i_1}, \tilde{x}_{2^t+i_2} \in \mathrm{aff}(\mathrm{L}_{i_k})$ and $\mathrm{S}_{t+1} \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}, \tilde{x}_{i_3}, \tilde{x}_{i_4}\}$, where $\tilde{x}_{i_3} := \tilde{x}_{2^t+i_1}$ and $\tilde{x}_{i_4} := \tilde{x}_{2^t+i_4}$. Because of the neighborhood relations between pseudovertices in S_t and S_{t+1} which we discussed in the proof of Theorem 4.2 we have $\dim(\mathrm{C}_{\mathrm{S}_{t+1}}(\tilde{x}_{i_j})|_{\mathrm{aff}(\mathrm{L}_{i_k})}) = 0$ for $j = 1, 2, 3, 4$, i.e. $\mathrm{C}_{\mathrm{S}_{t+1}}(\tilde{x}_{i_j})|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \tilde{x}_{i_j}$. It remains to be verified that for $\mathrm{Q} := \bigcap_{j=1}^{4} \mathrm{C}(\tilde{x}_{i_j})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ the inclusion $\mathrm{Q} \subseteq \mathrm{conv}(\bigcup_{j=1}^{4} \tilde{x}_{i_j})$ holds.

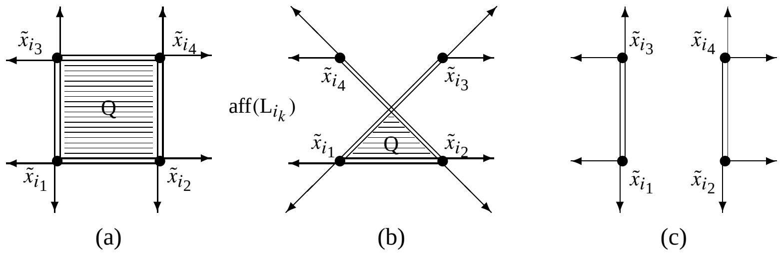

Figure 7. Examples of cones fulfilling Case 3 of Corollary 5.2.

It follows from the construction of CDP that we have either $Q = \emptyset$ or $\dim(Q) = 2$. In the former case we obviously have $Q \subseteq \text{conv}(\bigcup_{j=1}^{4} \tilde{x}_{i_j})$ (Figures 5(c), 7(c)). In the latter case Q is a pointed polyhedron and each facet of Q is 1-dimensional and contained in an edge of at least one of the cones $C(\tilde{x}_{i_j})|_{\text{aff}(L_{i_k})}$. However, each edge $E_{i_j,l}$ of the cone $C(\tilde{x}_{i_j})|_{\text{aff}(L_{i_k})}$ or its negative extension $E^-_{i_j,l}$ contains a pseudovertex $\tilde{x}_{i_g} \in S_{t+1} \cap \text{aff}(L_{i_k}) \setminus \{\tilde{x}_{i_j}\}$. Suppose that $\tilde{x}_{i_g} \in E^-_{i_j,l}$. Then $\tilde{x}_{i_j}$ and $\tilde{x}_{i_g}$ are N_3-neighbors and we have $E_{i_j,l} \cap Q = \emptyset$ (Figures 5(b), 7(b)). Suppose that $\tilde{x}_{i_g} \in E_{i_j,l}$. Then $\tilde{x}_{i_j}$ and $\tilde{x}_{i_g}$ are N_1-neighbors and we have $E_{i_j,l} \cap Q \subseteq \text{conv}(\tilde{x}_{i_l}, \tilde{x}_{i_g}\}$ (Figures 5(a), 7(a)). Hence, the facets of Q are contained in $\text{conv}(\bigcup_{j=1}^{4} \tilde{x}_{i_j})$ and by Corollary 3.1 we have $Q \subseteq \text{conv}(\bigcup_{j=1}^{4} \tilde{x}_{i_j})$. Thus for $S_{t+1} \cap \text{aff}(L_{i_k})$ Case 3 of Corollary 5.2 holds.

Case 3: Suppose that for $S_t \cap \text{aff}(L_{i_k})$ Case 3 of Corollary 5.2 holds, i.e. $\text{aff}(L_{i_k}) \cap S_t = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}, \tilde{x}_{i_3}, \tilde{x}_{i_4}\}$, $C_{S_t}(\tilde{x}_{i_j})|_{\text{aff}(L_{i_k})} = \tilde{x}_{i_j}$, and $\bigcap_{j=1}^{4} C(\tilde{x}_{i_j})|_{\text{aff}(L_{i_k})} \subseteq \text{conv}(\bigcup_{j=1}^{4} \tilde{x}_{i_j})$. Thus, $E_{i_j,1}$ and $E_{i_j,2}$ are not edges of $C(\tilde{x}_{i_j})|_{\text{aff}(L_{i_k})}$, i.e. $\bar{E}_{i_j} \neq E_{i_j,l}$ for $l = 1, 2$. By the construction of CDP and because of (28) we have $\tilde{x}_{2^t+i_j} \notin \text{aff}(L_{i_k})$ for $j = 1, 2, 3, 4$, and because of Corollory 5.1 we have $S_{t+1} \cap \text{aff}(L_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}, \tilde{x}_{i_3}, \tilde{x}_{i_4}\}$, i.e. for $S_{t+1} \cap \text{aff}(L_{i_k})$ case 3 of Corollary 5.2 holds. □

For an N-set S derived by CDP we can approximate the polyhedron P by the convex hull of the cones $C_S(\tilde{x}_1), C_S(\tilde{x}_2), \ldots, C_S(\tilde{x}_{2^t})$. To verify that an inequality $d^T x \geqslant \delta$ is a (P, Ω)-cut, in accordance with condition (B) of Theorem 4.1 we have to ensure that for every $\tilde{x}_i \in S$ with $d^T \tilde{x}_i < \delta$ this inequality eliminates only points in the portion of $C_S(\tilde{x}_i)$ contained in int(K). For a single cone $C_S(\tilde{x}_i)$ a convexity cut derived in the affine space spanned by $C_S(\tilde{x}_i)$ fulfills this condition. To fulfill condition (B) of Theorem 4.1 for the cones $C_S(\tilde{x}_1), C_S(\tilde{x}_2), \ldots, C_S(\tilde{x}_{2^t})$ simultaneously, the idea is to derive a cutting plane $d^T x \geqslant \delta$ that in the case of $d^T \tilde{x}_i < \delta$ is in the affine space spanned by $C_S(\tilde{x}_i)$ equivalent to a convexity cut derived w.r.t. $C_S(\tilde{x}_i)$. We shall see with the following proposition that such a cutting plane is a (P, Ω)-cut.

PROPOSITION 5.1. *Let* $S = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2^t}\}$ *be an N-set of* $V^{ps}(P_{(A,b)})$ *derived by CDP. For any* $i = 1, 2, \ldots, 2^t$ *and* $j = 1, 2, \ldots, n-t$ *let* $y_{i,j}(\tilde{\lambda}_{i,j})$ *be the intersection point of the edge* $E_{i,j} = \{y_{i,j}(\lambda) = \tilde{x}_i + \lambda \tilde{u}_{i,j} \mid \lambda \in \mathbf{R}_0^+\}$ *of the*

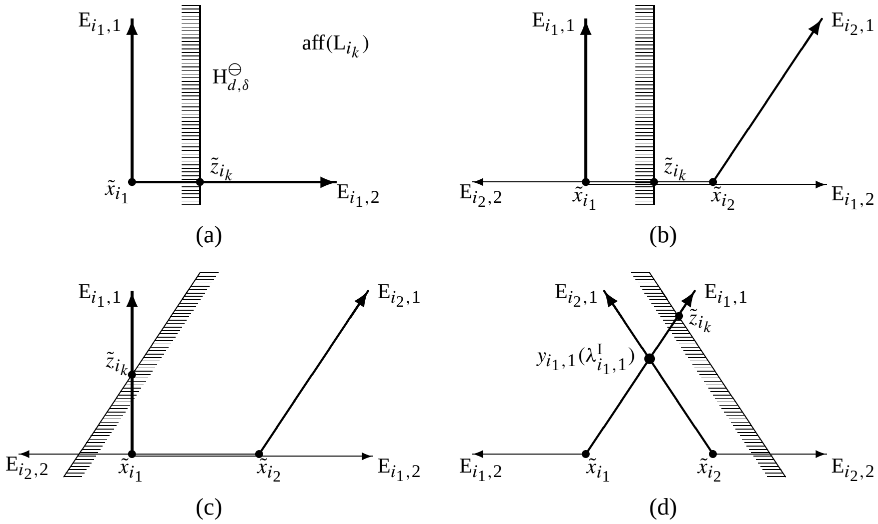

Figure 8. Cutting planes in aff(L_{i_k}) fulfilling the conditions of Proposition 5.1.

cone $C_S(\tilde{x}_i) = \tilde{x}_i + \text{cone}(\tilde{u}_{i,1}, \tilde{u}_{i,2}, \dots, \tilde{u}_{i,n-t})$ *and the boundary of* cl(K) *with the convention that* $\tilde{\lambda}_{i,j} = \infty$, $1/\tilde{\lambda}_{i,j} = 0$, *and* $y_{i,j}(\infty) = \emptyset$ *if such an intersection point does not exist. An inequality* $d^T x \geqslant \delta$ *fulfilling* $d^T\tilde{x}_i \neq \delta$ *and*

$$d^T\tilde{u}_{i,j} \geqslant \frac{1}{\tilde{\lambda}_{i,j}} \max\{(\delta - d^T\tilde{x}_i)\,,\, 0\} \quad \textit{for} \quad \begin{array}{l} i = 1, 2, \dots, 2^t \\ j = 1, 2, \dots, n-t \end{array} \tag{29}$$

is a (P, Ω)*-cut.*

Proof. Suppose that the inequality $d^T x \geqslant \delta$ fulfills the conditions of Proposition 5.1. For this inequality we have to verify the conditions of Theorem 4.1.

Let $\tilde{x}_i \in S$ such that $d^T\tilde{x}_i > \delta$. It follows from (29) that $d^T\tilde{u}_{i,j} \geqslant 0$ for $j = 1, 2, \dots, n-t$. Hence, we have $C_S(\tilde{x}_i) \subseteq H^{\oplus}_{d,\delta}$ for all $\tilde{x}_i \in S \cap H^{\oplus}_{d,\delta}$. Since condition (A) of Theorem 4.1 is fulfilled by assumption, it remains to verifiy conditions (B) and (C).

For $\tilde{x}_i \in S$ with $d^T\tilde{x}_i < \delta$, the inequality (29) can be written as $d^T\tilde{u}_{i,j} \geqslant 0$ in the case of $\tilde{\lambda}_{i,j} = \infty$, and as $d^T y_{i,j}(\tilde{\lambda}_{i,j}) \geqslant \delta$ otherwise. Hence in the affine space spanned by $C_S(\tilde{x}_i)$ the inequality $d^T x \geqslant \delta$ is equivalent to a convexity cut derived w.r.t. the cone $C_S(\tilde{x}_i)$, i.e. $d^T x \geqslant \delta$ eliminates with $\tilde{x}_i$ only points in the portion of $C_S(\tilde{x}_i)$ contained in int(K). Therefore, $d^T x \geqslant \delta$ fulfills condition (B) of Theorem 4.1.

To verify condition (C) we consider a vector $\tilde{r}_k$, which is derived according to Theorem 3.2. Let $\tilde{x}_{i_k} \in S \cap H^{-}_{d,\delta}$, and let L_{i_k} be a face of $C(\tilde{x}_{i_k})$ with $\dim(L_{i_k}) = 2$ such that with $\tilde{x}_{i_k}$, L_{i_k} and

$$Q_k := \bigcap_{\tilde{x}_{i_j} \in S \cap \text{aff}(L_{i_k})} C(\tilde{x}_{i_j})|_{\text{aff}(L_{i_k})} \tag{30}$$

$\tilde{r}_k$ fulfills the conditions of Theorem 3.2. We have $\mathrm{C}(\tilde{x}_{i_k})|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \mathrm{L}_{i_k}$ and by (27) we have

$$\mathrm{C}(\tilde{x}_{i_j})|_{\mathrm{aff}(\mathrm{L}_{i_k})} = \tilde{x}_{i_j} + \mathrm{cone}(\tilde{u}_{i_j,1}, \tilde{u}_{i_j,2}) \quad \forall\, \tilde{x}_{i_j} \in \mathrm{S} \cap \mathrm{aff}(\mathrm{L}_{i_k}).$$

Let $\mathrm{E}_{i_j,l} = \{y_{i_j,l}(\lambda) = \tilde{x}_{i_j} + \lambda \tilde{u}_{i_j,l} \mid \lambda \in \mathbf{R}_0^+\}$ and $\mathrm{E}^-_{i_j,l} = \{y_{i_j,l}(\lambda) = \tilde{x}_{i_j} + \lambda \tilde{u}_{i_j,l} \mid \lambda \in \mathbf{R}_0^-\}$, and to simplify notation let $\tilde{x}_{i_1} := \tilde{x}_{i_k}$.

Since $\mathrm{Q}_k \cap \mathrm{H}^{\ominus}_{d,\delta} \neq \mathrm{Q}_k$, $\dim(\mathrm{Q}_k \cap \mathrm{H}^{\ominus}_{d,\delta}) = 2$ and $\tilde{x}_{i_1} \in \mathrm{S} \cap \mathrm{H}^-_{d,\delta}$ (cf. Theorem 3.2), the hyperplane $\mathrm{H}_{d,\delta}$ intersects at least one edge of the cone $\mathrm{C}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ at a point $\tilde{z}_{i_k}$ different from $\tilde{x}_{i_1}$ (Figure 8(a)). However, since by assumption $\mathrm{Q}_k \cap \mathrm{H}_{d,\delta}$ is a half-line (cf. Theorem 3.2), it follows from the definition of Q_k that $\mathrm{H}_{d,\delta}$ intersects one and only one edge of $\mathrm{C}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$. For the same reason the polyhedron Q_k has to be unbounded. Of the alternatives for $\mathrm{S} \cap \mathrm{aff}(\mathrm{L}_{i_k})$ described in Corollary 5.2 there are only two for which Q_k can be unbounded.

Case 1: Suppose that $\mathrm{S} \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}\}$, i.e. $\mathrm{Q}_k = \mathrm{C}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ (Figure 8(a)). Then $\mathrm{E}_{i_1,1}$ and $\mathrm{E}_{i_1,2}$ are also edges of $\mathrm{C}_{\mathrm{S}}(\tilde{x}_{i_1})$ (see (28)). In the first variant the hyperplane $\mathrm{H}_{d,\delta}$ intersects $\mathrm{E}_{i_1,2}$ and does not intersect $\mathrm{E}_{i_1,1}$. Since $d^T\tilde{x}_{i_1} < \delta$, we have $d^T\tilde{u}_{i_1,1} \leqslant 0$. On the other hand, because of $\tilde{\lambda}_{i_1,1} > 0$ and $\delta - d^T\tilde{x}_{i_1} > 0$ by (29) we have $d^T\tilde{u}_{i_1,1} \geqslant 1/\tilde{\lambda}_{i_1,1}(\delta - d^T\tilde{x}_{i_1}) \geqslant 0$. This implies that $d^T\tilde{u}_{i_1,1} = 0$, i.e. $\tilde{r}_k = \tilde{u}_{i_1,1}/\|\tilde{u}_{i_1,1}\|$. Furthermore, since $\delta - d^T\tilde{x}_{i_1} > 0$, we have $1/\tilde{\lambda}_{i_1,1} = 0$. However, $1/\tilde{\lambda}_{i_1,1} = 0$ is equivalent to $\mathrm{E}_{i_1,1} \subseteq \mathrm{int}(\mathrm{K})$, which implies $x + \lambda\tilde{r}_k \in \mathrm{int}(\mathrm{K})$ for all $x \in \mathrm{int}(\mathrm{K})$, $\lambda \in \mathbf{R}_0^+$. A similar argument can be used for the second variant in which $\mathrm{H}_{d,\delta}$ intersects $\mathrm{E}_{i_1,1}$.

Case 2: Suppose that we have $\mathrm{S} \cap \mathrm{aff}(\mathrm{L}_{i_k}) = \{\tilde{x}_{i_1}, \tilde{x}_{i_2}\}$ where $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$ are neighbors. Hence, we have $\mathrm{Q}_k = \mathrm{C}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})} \cap \mathrm{C}(\tilde{x}_{i_2})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$. Since $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$ are nondegenerate we can assume w.l.o.g. $\tilde{x}_{i_2} \in \mathrm{E}_{i_1,2} \cup \mathrm{E}^-_{i_1,2}$ and $\tilde{x}_{i_1} \in \mathrm{E}_{i_2,2} \cup \mathrm{E}^-_{i_2,2}$. Hence, $\mathrm{E}_{i_1,1}$ and $\mathrm{E}_{i_2,1}$ are also edges of the cones $\mathrm{C}_{\mathrm{S}}(\tilde{x}_{i_1})$ and $\mathrm{C}_{\mathrm{S}}(\tilde{x}_{i_2})$. In Figure 8 the edges of $\mathrm{C}(\tilde{x}_{i_1})$ and $\mathrm{C}(\tilde{x}_{i_2})$ that are also edges of $\mathrm{C}_{\mathrm{S}}(\tilde{x}_{i_1})$ and $\mathrm{C}_{\mathrm{S}}(\tilde{x}_{i_2})$ are indicated by thick lines. Note that by the construction of CDP there always exists a constraint $a_s^T x \leqslant \beta_s$ of $Ax \leqslant b$ such that $\mathrm{E}_{i_1,2} \cup \mathrm{E}^-_{i_1,2} = \mathrm{E}_{i_2,2} \cup \mathrm{E}^-_{i_2,2} = \{x \in \mathbf{R}^n \mid a_s^T x = \beta_s\} \cap \mathrm{aff}(\mathrm{L}_{i_k})$ and $\mathrm{E}_{i_1,1}, \mathrm{E}_{i_2,1} \subseteq \{x \in \mathbf{R}^n \mid a_s^T x \leqslant \beta_s\} \cap \mathrm{aff}(\mathrm{L}_{i_k})$ (cf. condition (B) of Theorem 4.2 and Lemma 4.1). Since the hyperplane $\mathrm{H}_{d,\delta}$ intersects one and only one edge of $\mathrm{C}(\tilde{x}_{i_1})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$, we have to consider the following alternatives.

(a) Suppose that the hyperplane $\mathrm{H}_{d,\delta}$ intersects $\mathrm{E}_{i_1,2}$. Then it does not intersect $\mathrm{E}_{i_1,1}$ and we can verify that $\tilde{r}_k = \tilde{u}_{i_1,1}/\|\tilde{u}_{i_1,1}\|$ with $x + \lambda\tilde{r}_k \subseteq \mathrm{int}(\mathrm{K})$ for all $x \in \mathrm{int}(\mathrm{K})$, $\lambda \in \mathbf{R}_0^+$ as in case 1 (Figure 8(b)).

(b) Suppose that the hyperplane $\mathrm{H}_{d,\delta}$ intersects $\mathrm{E}_{i_1,1}$. Then it does not intersect $\mathrm{E}_{i_1,2}$. We now have to distinguish between the N_1- and N_3-neighorhood of $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$.

Suppose that $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$ are N_1-neighbors, i.e. $\tilde{x}_{i_1} \in \mathrm{E}_{i_2,2}$ and $\tilde{x}_{i_2} \in \mathrm{E}_{i_1,2}$ (Figure 8(c)). Since $\mathrm{H}_{d,\delta}$ does not intersect $\mathrm{E}_{i_1,2}$, we have $d^T\tilde{x}_{i_2} < \delta$. The hyperplane $\mathrm{H}_{d,\delta}$ also does not intersect $\mathrm{E}_{i_2,1}$ of $\mathrm{C}(\tilde{x}_{i_2})|_{\mathrm{aff}(\mathrm{L}_{i_k})}$ because otherwise $\mathrm{Q}_k \cap \mathrm{H}_{d,\delta}$ is bounded. This implies that $d^T\tilde{u}_{i_2,1} \leqslant 0$. However, $\mathrm{E}_{i_2,1}$ is an edge of $\mathrm{C}_{\mathrm{S}}(\tilde{x}_{i_2})$. Because of

$\tilde{\lambda}_{i_2,1} > 0$ and $\delta - d^T\tilde{x}_{i_2} > 0$ by (29) we therefore have $d^T\tilde{u}_{i_2,1} \geqslant 1/\tilde{\lambda}_{i_2,1}(\delta - d^T\tilde{x}_{i_2}) \geqslant 0$. This implies $d^T\tilde{u}_{i_2,1} = 0$ and $1/\tilde{\lambda}_{i_2,1} = 0$. Thus we have $\tilde{r}_k = \tilde{u}_{i_2,1}/\|\tilde{u}_{i_2,1}\|$ and $x + \lambda\tilde{r}_k \in \text{int(K)}$ for all $x \in \text{int(K)}$, $\lambda \in \mathbf{R}_0^+$ (see Case 1).

Suppose that $\tilde{x}_{i_1}$ and $\tilde{x}_{i_2}$ are N_3-neighbors, i.e. $\tilde{x}_{i_1} \in \text{E}^-_{i_2,2}$ and $\tilde{x}_{i_2} \in \text{E}^-_{i_1,2}$ (Figure 8(d)). Since $\dim(\text{Q}_k) = 2$, there exists $\lambda^{\text{I}}_{i_1,1} \in \mathbf{R}^+$ such that

$$\text{Q}_k = y_{i_1,1}(\lambda^{\text{I}}_{i_1,1}) + \text{cone}(\tilde{u}_{i_1,1}, \tilde{u}_{i_2,1}).$$

We claim $d^T\tilde{x}_{i_2} < \delta$. Indeed, let us assume the contrary, i.e. $d^T\tilde{x}_{i_2} > \delta$ (we have by assumption $d^T\tilde{x}_{i_2} \neq \delta$). It follows from the condition $\text{C}_\text{S}(\tilde{x}_i) \subseteq \text{H}^{\oplus}_{d,\delta}$ for all $\tilde{x}_i \in \text{S} \cap \text{H}^{\oplus}_{d,\delta}$ of Theorem 3.2 that $\text{C}(\tilde{x}_i)|_{\text{aff}(\text{L}_{i_k})} \subseteq \text{H}^+_{d,\delta}$ for all $\tilde{x}_i \in \text{S} \cap \text{H}^+_{d,\delta}$. Hence, we have $\text{C}(\tilde{x}_{i_2})|_{\text{aff}(\text{L}_{i_k})} \subseteq \text{H}^+_{d,\delta}$ which contradicts $\dim(\text{Q}_k \cap \text{H}^{\ominus}_{d,\delta}) = 2$ (cf. (30)). Thus, $d^T\tilde{x}_{i_2} < \delta$. Because of $\dim(\text{Q}_k \cap \text{H}^{\ominus}_{d,\delta}) = 2$ we also have $d^Ty_{i_1,1}(\lambda^{\text{I}}_{i_1,1}) < \delta$. Furthermore, since $\text{Q}_k \cap \text{H}_{d,\delta}$ is unbounded, $\text{H}_{d,\delta}$ does not intersect the ray $\{y_{i_1,1}(\lambda^{\text{I}}_{i_1,1}) + \lambda\tilde{u}_{i_2,1} \mid \lambda \in \mathbf{R}_0^+\}$. Hence, we have $d^T\tilde{u}_{i_2,1} \leqslant 0$. Thus, with the same argument as above we can verify that $d^T\tilde{u}_{i_2,1} = 0$ and $1/\tilde{\lambda}_{i_2,1} = 0$ such that $\tilde{r}_k = \tilde{u}_{i_2,1}/\|\tilde{u}_{i_2,1}\|$ and $x + \lambda\tilde{r}_k \subseteq \text{int(K)}$ for all $x \in \text{int(K)}$, $\lambda \in \mathbf{R}_0^+$.

We have already verified $x + \lambda\tilde{r}_k \subseteq \text{int(K)}$ for all $x \in \text{int(K)}$, $\lambda \in \mathbf{R}_0^+$. Since $\tilde{r}_k$ was chosen arbitrarily, we have $x + \text{cone}(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_t) \subseteq \text{int(K)}$ for all $x \in \text{int(K)}$, which verifies condition (C) of Theorem 4.1. □

In Proposition 5.1 we specified the conditions of Theorem 4.1 for the N-sets derived by CDP. An inequality fulfilling Proposition 5.1 is referred to as a *decomposition cut*. We now have to examine the existence of a decomposition cut.

LEMMA 5.1. *Let* $\text{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2^t}\}$ *be an N-set derived by CDP. There always exists a cutting plane* $d^Tx \geqslant \delta$ *with* $d^T\tilde{x}_i < \delta$ *for all* $\tilde{x}_i \in \text{S}$, *which fulfills the conditions of Proposition 5.1.*

Proof. By construction of CDP for $\text{P} = \{x \in \mathbf{R}^n \mid Ax \leqslant b\}$ and the N-set $\text{S} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{2^t}\}$ there exist $n - t$ constraints of $Ax \leqslant b$, namely $a_1^Tx \leqslant \beta_1$, $a_2^Tx \leqslant \beta_2, \ldots, a_{n-t}^Tx \leqslant \beta_{n-t}$ such that for $i = 1, 2, \ldots, 2^t$ and $j = 1, 2, \ldots, n - t$ the following hold.

1. $a_1^T\tilde{x}_i = \beta_1$, $a_2^T\tilde{x}_i = \beta_2, \ldots, a_{n-t}^T\tilde{x}_i = \beta_{n-t}$;
2. If $0 < \tilde{\lambda}_{i,j} < \infty$, then $a_1^Ty_{i,j}(\tilde{\lambda}_{i,j}) = \beta_1, \ldots, a_{j-1}^Ty_{i,j}(\tilde{\lambda}_{i,j}) = \beta_{j-1}$, $a_j^Ty_{i,j}(\tilde{\lambda}_{i,j}) < \beta_j$, $a_{j+1}^Ty_{i,j}(\tilde{\lambda}_{i,j}) = \beta_{j+1}, \ldots, a_{n-t}^Ty_{i,j}(\tilde{\lambda}_{i,j}) = \beta_{n-t}$;
3. If $\tilde{\lambda}_{i,j} = \infty$, then $a_1^T\tilde{u}_{i,j} = 0, \ldots, a_{j-1}^T\tilde{u}_{i,j} = 0$, $a_j^T\tilde{u}_{i,j} \leqslant 0$, $a_{j+1}^T\tilde{u}_{i,j} = 0, \ldots, a_{n-t}^T\tilde{u}_{i,j} = 0$;

(cf. Lemma 4.1). By defining

$$d := -\sum_{j=1}^{n-t} a_j \quad \text{and} \quad \delta := \min_{i=1}^{2^t}\min_{j=1}^{n-t}\{\, d^Ty_{i,j}(\tilde{\lambda}_{i,j}) \mid 0 < \tilde{\lambda}_{i,j} < \infty\} \tag{31}$$

we get $d^T\tilde{x}_i < \delta$. Furthermore, $d^T y_{i,j}(\tilde{\lambda}_{i,j}) \geqslant \delta$ if $0 < \tilde{\lambda}_{i,j} < \infty$, and $d^T\tilde{u}_{i,j} \geqslant 0$ otherwise. This is equivalent to $d^T\tilde{u}_{i,j} \geqslant 1/\tilde{\lambda}_{i,j}(\delta - d^T\tilde{x}_i) = 1/\tilde{\lambda}_{i,j}\max\{(\delta - d^T\tilde{x}_i), 0\}$. □

In Proposition 5.1 we gave some conditions to verify that a cutting plane $d^Tx \geqslant \delta$ is a (P, Ω)-cut. Conversely, we can utilize these conditions to derive a (P, Ω)-cut by choosing in advance a set $S^< \subseteq S$ of pseudovertices that shall be eliminated. By doing so, the conditions of Proposition 5.1 can be written as a system of inequalities. Every feasible solution (d, δ) of these inequalites yields a (P, Ω)-cut $d^Tx \geqslant \delta$.

Our aim is to derive a deep cutting plane, i.e. a cutting plane that eliminates as much of each cone $C_S(\tilde{x}_i) = \tilde{x}_i + \text{cone}(\tilde{u}_{i,1}, \tilde{u}_{i,2}, \ldots, \tilde{u}_{i,n-t})$ with $\tilde{x}_i \in S^<$ as possible. For this we have to define a (heuristic) measure for the depth of such a cutting plane.

Setting

$$\bar{u}_i := \frac{1}{n-t}\sum_{j=1}^{n-t}\frac{\tilde{u}_{i,j}}{\|\tilde{u}_{i,j}\|},$$

we have an average direction of the edges of the cone $C_S(\tilde{x}_i)$ such that the half-line $E_i^{avg} = \{\bar{y}_i(\lambda) = \tilde{x}_i + \lambda\bar{u}_i \mid \lambda \in \mathbf{R}_0^+\}$ is contained in $C_S(\tilde{x}_i)$. Let $d^Tx \geqslant \delta$ be a cutting plane such that $d^T\tilde{x}_i < \delta$ and suppose that $d^Tx = \delta$ intersects E_i^{avg} at $\bar{y}_i(\Delta)$, where $\Delta \in \mathbf{R}_0^+$. In general it hold that the larger Δ is, i.e. the larger the distance from $\bar{y}_i(\Delta)$ to $\tilde{x}_i$, the larger the portion of $C_S(\tilde{x}_i)$ that is eliminated by the cutting plane $d^Tx \geqslant \delta$ usually turns out to be.

Since a cutting plane shall eliminate as much as possible of each of the cones $C_S(\tilde{x}_i)$ with $\tilde{x}_i \in S$ simultaneously, this leads to the following heuristic measure of the depth of a cutting plane, where $\mathcal{J} := \{i \in \{1, 2, \ldots, 2^t\} \mid \tilde{x}_i \in S^<\}$. Setting

$$\bar{x} := \frac{1}{|\mathcal{J}|}\sum_{i\in\mathcal{J}}\tilde{x}_i \quad \text{and} \quad \bar{v} := \frac{1}{|\mathcal{J}|}\sum_{i\in\mathcal{J}}\bar{u}_i,$$

we define the depth of a cutting plane $d^Tx \geqslant \delta$ by a measure $\Delta(d, \delta)$:

$$\Delta(d,\delta) := \begin{cases} \dfrac{\delta - d^T\bar{x}}{d^T\bar{v}} & \text{for } d^T\bar{v} > 0 \\ \infty & \text{otherwise} \end{cases}.$$

By definition of $\Delta(d, \delta)$ we have $d^T(\bar{x} + \Delta(d, \delta)\bar{v}) = \delta$. $\Delta(d, \delta)$ can be interpreted as a measure for the average depth of the cutting plane $d^Tx \geqslant \delta$ with respect to each of the cones $C_S(\tilde{x}_i)$ with $\tilde{x}_i \in S^<$. Therefore, the larger $\Delta(d, \delta)$ is, the deeper the cutting plane $d^Tx \geqslant \delta$ usually is. Let us consider the following minimization

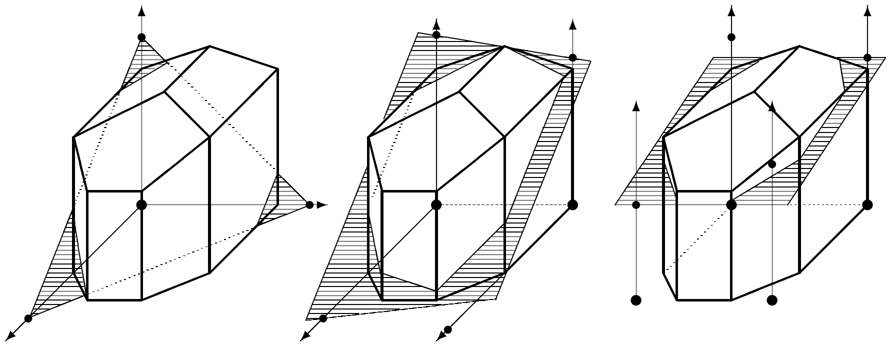

Figure 9. Decomposition cuts derived w.r.t. different decomposition depths.

problem.

$$\begin{array}{lrll} \text{minimize} & d^T\bar{v} & & \\ \text{subject to} & d^T\bar{v} & \geqslant \varrho & \\ & -d^T\bar{x}+\delta & = 1 & \\ & d^T\tilde{x}_i-\delta & \leqslant -\varrho & \text{for } \tilde{x}_i \in \mathrm{S}^< \\ & d^T\tilde{x}_i-\delta & \geqslant \varrho & \text{for } \tilde{x}_i \in \mathrm{S}\backslash\mathrm{S}^< \\ & d^T y_{i,j}(\tilde{\lambda}_{i,j})-\delta & \geqslant 0 & \text{for } \tilde{x}_i \in \mathrm{S}^< \text{ and } 0<\tilde{\lambda}_{i,j}<\infty \\ & d^T\tilde{u}_{i,j} & \geqslant 0 & \text{for } \tilde{x}_i \in \mathrm{S}^< \text{ and } \tilde{\lambda}_{i,j}=\infty \\ & d^T\tilde{u}_{i,j} & \geqslant 0 & \text{for } \tilde{x}_i \in \mathrm{S}\backslash\mathrm{S}^< \end{array} \qquad (32)$$

where $\varrho \in \mathbf{R}^+$ is sufficiently small. By solving (32) we get a cutting plane $d^Tx \geqslant \delta$ with the depth $\Delta(d,\delta) = 1/d^T\bar{v}$ that fulfills the conditions of Proposition 5.1. If no solution of (32) exists, we have to augment the set $\mathrm{S}^<$. Note that for a small ϱ the solvability of (32) is ensured with $\mathrm{S}^< = \mathrm{S}$ by Lemma 5.1.

EXAMPLE 5.1. In Figure 9 decomposition cuts are indicated that are derived w.r.t. N-sets obtained at different stages of CDP (see Example 4.2). The decomposition cut in Figure 9(a), which is derived w.r.t $\mathrm{S}_0 := \{\tilde{x}_1 = x_0\}$, is equivalent to the x_0-eliminating intersection cut. We can see that by increasing decomposition depth the decomposition cut eliminates a larger portion of $\mathrm{P} \cap \mathrm{int}(\mathrm{K})$.

6. Numerical Experiments

To compare the performance of decomposition cuts with the performance of convexity cuts, we applied both types of cuts to pure cutting plane algorithms for concave minimization. A concave minimization problem is as follows:

$$\min\{f(x) \mid x \in \mathrm{P}\}, \qquad (33)$$

where $f : \mathbf{R}^n \mapsto \mathbf{R}$ is a concave function and P is a full-dimensional polytope in $\mathbf{R}^n$. It is well-known that there exists a vertex of P, which is a global optimum. Hence, we can restrict our search to the set of vertices V(P) of P. Accordingly, $x_0 \in$ V(P) is said to be a local star optimum if $f(x_0) \leqslant f(x)$ for all $x \in$ V(P) adjacent to x_0. Often it suffices to find an ϵ-optimal solution ($\epsilon \in \mathbf{R}^+$), where $\hat{x} \in$ P is said to be ϵ-optimal if $f(\hat{x}) \leqslant f(x) + \epsilon$ for all $x \in$ P. A cutting plane algorithm to determine an ϵ-optimal solution of (33) consists of two main steps.

Initialization: Set $P_0 := P$, $\hat{f} := \infty$, $\Omega := \{x \in \mathbf{R}^n \mid f(x) < \hat{f} - \epsilon\}$, $i := 0$.

Step 1: Find a local star optimum $x_0^i \in V(P_i)$. If $f(x_0^i) < \hat{f}$, then set $\hat{f} := f(x_0^i)$ and $\hat{x} := x_0^i$. Go to Step 2.

Step 2: Derive a (P_i, Ω)-cut $h_i^T x \geqslant \theta_i$ such that $h_i^T x_0^i < \theta_i$, and set $P_{i+1} := P_i \cap \{x \in \mathbf{R}^n \mid h_i^T x \geqslant \theta_i\}$. If $P_{i+1} = \emptyset$, then $\hat{x}$ is an ϵ-optimal solution, otherwise set $i := i + 1$ and return to Step 1.

Based on this scheme we constructed an algorithm using intersection cuts (cf. Section 1), termed the Intersection Cut Algorithm (ICA), and an algorithm using decomposition cuts, the Decomposition Cut Algorithm (DCA). In these algorithms Step 1 and Step 2 were performed as follows.

Step 1: First determine a vertex $x_0 \in V(P_i)$ by solving $\min\{c_i^T x \mid x \in P_i\}$, where $c_i \in [-10, 10]^n$ is a uniformly distributed random vector. Starting with $j := 0$, examine the vertices adjacent to x_j and determine from among them the vertex x_{j+1} with the smallest objective value. If $f(x_{j+1}) < f(x_j)$, then set $j := j + 1$ and repeat this process, otherwise set $x_0^i := x_j$.

Step 2: Since $f(x)$ is concave, the set $K = \{x \in \mathbf{R}^n \mid f(x) \geqslant \hat{f} - \epsilon\}$ is convex. We have $\text{int}(K) \cap (P_i \cap \Omega) = \emptyset$ and $x_0^i \in \text{int}(K)$. To eliminate the nondegenerate vertex $x_0^i \in V(P_i)$ in

ICA we derive an intersection cut w.r.t. K and P_i, and in

DCA we start at K and P_i CDP with a maximal decomposition depth of level 3, and derive with the resulting N-set S a decomposition cut by solving (32) in which $S^< := S$.

If in DCA there exist two or more N-isomorph sets each of which fulfills the if-conditions of CDP, we have to choose an appropriate candidate. For this purpose we applied the following *N-isomorph-set rule*, where $M \in \mathbf{R}^+$ is a sufficiently large constant.

N-isomorph-set rule: Let $R_{S_t} = \{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_{2^t}\}$ be an N-isomorph set, and let $\eta_{i,j_i} \in \mathbf{R}^+$ be chosen so that in the case of $\bar{E}_i \not\subseteq \text{cl}(K)$, $y_{i,j_i}(\eta_{i,j_i})$ is the point of intersection of the cone edge $\bar{E}_i = \{y_{i,j_i}(\lambda) = \tilde{x}_i + \lambda \tilde{u}_{i,j_i} \mid \lambda \in \mathbf{R}_0^+\}$ and the boundary of cl(K), and $\eta_{i,j_i} = M$ otherwise. Define $\bar{x} = \frac{1}{2^t} \sum_{i=1}^{2^t} \tilde{x}_i$

and $z(\mathrm{R}_{\mathrm{S}_t}) = \frac{1}{2^t}\sum_{i=1}^{2^t} y_{i,j_i}(\eta_{i,j_i})$. From all N-isomorph sets fulfilling the if-conditions we choose the one for which $\|\bar{x} - z(\mathrm{R}_{\mathrm{S}_t})\|$ is minimal.

For a chosen N-isomorph set $\mathrm{R}_{\mathrm{S}_t}$ there may also exist more than one constraint $a_{l_t^*}^T x \leqslant \beta_{l_t^*}$ fulfilling the if-conditions of CDP. In this case we applied the following *constraint rule*, where η_{i,j_i} are defined as in the N-isomorph-set rule.

Constraint rule: If $\bar{\mathrm{E}}_i$ intersects the hyperplane $a_{l_s}^T x = \beta_{l_s}$, then determine $\xi_{i,j_i}(l_s) \in \mathbf{R}^+$ such that $y_{i,j_i}(\xi_{i,j_i}(l_s))$ is the point of intersection of $\bar{\mathrm{E}}_i$ and $a_{l_s}^T x = \beta_{l_s}$. Otherwise set $\xi_{i,j_i}(l_s) = \eta_{i,j_i}$. Define $d(l_s) = \max_{i=1}^{2^t} \|y_{i,j_i}(\xi_{i,j_i}(l_s)) - y_{i,j_i}(\eta_{i,j_i})\|$. From all constraints $a_{l_1}^T x \leqslant \beta_{l_1}, a_{l_2}^T x \leqslant \beta_{l_2}, \ldots, a_{l_{h(l)}}^T x \leqslant \beta_{l_{h(l)}}$ fulfilling the if-conditions for $\mathrm{R}_{\mathrm{S}_t}$, we choose the one maximizing $d(l_s)$.

The algorithms were coded in Pascal 7.0 and run on a Pentium-90 PC. To compare the performance of the algorithms, we applied them to concave minimization problems of the form

$$\min\{f_s(x) \mid A_n x \leqslant b_n, x \geqslant 0\},$$

where

$$A_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ n & 1 & \cdots & n-2 & n-1 \end{pmatrix} \quad \text{and} \quad b_n = \frac{n(n+1)}{2}e,$$

and e is a vector of n ones. The values of the functions $f_s : \mathbf{R}^n \mapsto \mathbf{R}$, $s = 1, 2, 3$, are defined at $x = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n$ by

$$f_1(x) = \xi_1 + \tfrac{1}{2}\xi_2 + \cdots + \tfrac{1}{n}\xi_n - \sqrt{\xi_1^2 + 2\xi_2^2 + \cdots + n\xi_n^2};$$

$$f_2(x) = -(\xi_1^2 + 2\xi_2^2 + \cdots + n\xi_n^2) \cdot \ln(1 + \xi_1^2 + \cdots + \xi_n^2);$$

$$f_3(x) = -\max_{i=1}^{3} \|x - y_i\|_2 \quad \text{with} \quad \begin{array}{rcl} y_1 &=& 0,42 \cdot (1, 2, \ldots, n)^T \\ y_2 &=& 0,5 \cdot e \\ y_3 &=& 0,3 \cdot (n-1, n-2, \ldots, 0)^T. \end{array}$$

The system $A_n x \leqslant b_n$ is taken from Konno [12] and the functions $f_s : \mathbf{R}^n \mapsto \mathbf{R}$ are modifications of concave functions, which can be found in Horst et al. [10].

The test problems are very difficult to solve by cutting plane algorithms. They were chosen, because they provide typical examples of the performance of ICA and DCA in a very compact way.

We searched only for ϵ-optimal solutions, where the respective ϵ were chosen such that the objective value of an ϵ-optimal solution differed from the optimal value by 1% at most. Because the search for a local star optimum contains stochastic elements, we used the cutting plane algorithms 50 times for each test problem.

Table 1. Computational results of ICA and DCA.

		ICA				DCA			
		Average		Mean variation		Average		Mean variation	
fct.	n	cuts	time s	cuts	time s	cuts	time s	cuts	time s
f_1	6	2.5	0.39	0.7	0.10	1.7	1.22	1.1	0.67
f_1	7	3.6	0.75	0.7	0.21	1.7	1.68	1.2	1.02
f_1	8	5.8	1.65	0.9	0.28	1.8	2.32	1.7	1.85
f_1	9	9.5	3.85	1.5	0.70	2.3	4.07	1.6	2.41
f_1	10	21.7	14.24	4.2	3.59	2.7	6.59	2.5	5.11
f_1	11	83.5	135.19	25.9	66.23	2.8	8.64	5.1	14.39
f_1	12	–	–	–	–	9.5	39.29	14.9	63.49
f_1	13	–	–	–	–	12.0	66.25	21.7	125.35
f_2	6	16.4	3.50	4.9	1.38	2.7	2.02	0.4	0.22
f_2	7	42.5	16.67	7.6	4.63	2.7	2.64	0.5	0.46
f_2	8	244.9	467.24	42.9	162.18	4.2	5.35	2.0	2.43
f_2	9	–	–	–	–	6.6	11.24	4.9	11.24
f_2	10	–	–	–	–	10.1	21.23	6.0	13.36
f_2	11	–	–	–	–	72.8	269.90	20.0	108.38
f_3	6	7.1	1.38	0.8	0.19	1.1	1.04	0.3	0.29
f_3	7	9.1	2.35	0.5	0.23	2.0	2.55	0.0	0.17
f_3	8	17.3	7.07	3.3	1.82	2.0	3.46	0.0	0.05
f_3	9	38.8	29.10	16.6	17.83	3.3	6.35	0.5	0.75
f_3	10	162.1	391.81	66.9	250.30	4.6	9.57	0.5	1.04
f_3	11	–	–	–	–	9.2	27.02	5.9	20.52
f_3	12	–	–	–	–	21.3	92.54	19.1	105.34
f_3	13	–	–	–	–	48.1	282.58	25.0	169.91

From the 40 fastest results we calculated the average number of cutting planes needed (cuts), the average time needed in seconds (time s.), and the respective mean variations. The results of the tests are shown in Table I, where a hyphen indicates, that the algorithm derived more than 400 cutting planes for at least 10 out of the 50 tests.

Both ICA and DCA are very sensitive to modifications. For example, by replacing the above procedure for searching for a local star optimum by Zwart's Procedure II (cf. Zwart [24]), in both algorithms the number of cutting planes required increased by up to 50%. Similar observations were made when the N-isomorph-set and constraint rule in CDP were modified. The following example may help to explain the differences in performance of ICA and DCA.

EXAMPLE 6.1. Let us consider the concave minimization problem

$$\min\{-x^T Ex + e^T x \mid 0 \leqslant x \leqslant e\}, \tag{34}$$

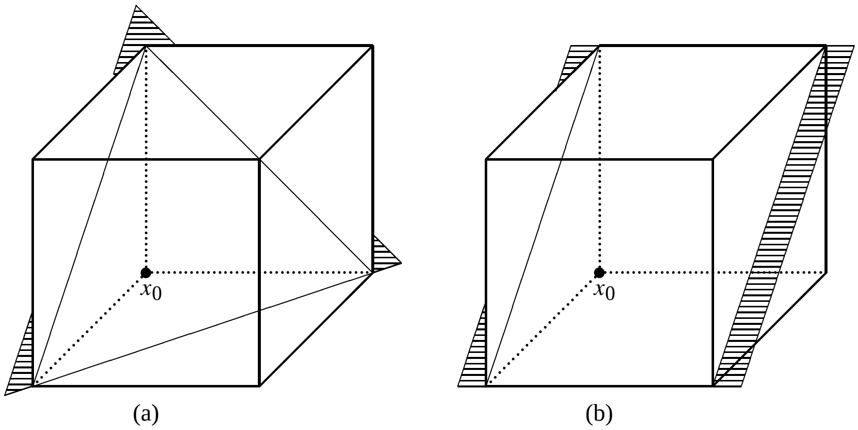

Figure 10. An intersection cut and a corresponding decomposition cut of level 1.

where $E = \mathrm{diag}(1, 1, \ldots, 1)$ denotes the unit matrix and e is a vector of n ones. Each vertex of the unit hypercube $\mathrm{W} = \{x \in \mathbf{R}^n \mid 0 \leqslant x \leqslant e\}$ is a global optimum of (34) with objective value 0. Let x_0 be an arbitrary vertex of W and let the convex set K defined by $\mathrm{K} := \{x \in \mathbf{R}^n \mid -x^T E x + e^T x \geqslant -\epsilon\}$, where $\epsilon > 0$ is a prechosen tolerance.

Let V_n^I be the portion of polyhedron volume removed by a x_0-eliminating intersection cut and let $\mathrm{V}_n^{D_t}$ be the portion of polyhedron volume removed by the corresponding decomposition cut, where t denotes the level of decomposition depth in CDP (see Figures 10(a), 10(b)). For small ϵ we have $\mathrm{V}_n^I \approx \frac{1}{n!}$ and $\mathrm{V}_n^{D_t} \approx \frac{1}{(n-t)!}$, i.e. $\mathrm{V}_n^{D_t} \approx n \cdot (n-1) \cdot \ldots \cdot (n-t+1) \cdot \mathrm{V}_n^I$. Thus for $t = 3$, a decomposition cut removes a polyhedron volume of W which is approximately $n(n-1)(n-2)$ times the polyhedron volume of W that is removed by an intersection cut.

According to Example 6.1 decomposition cuts become with increasing dimension more and more superior to intersection cuts. Furthermore, we can see that with increasing dimension the benefit of a further cone decomposition in CDP also increases. Based on the numerical experiments this leads us to assume, that in algorithms which use convexity cuts, the replacement of convexity cuts by decomposition cuts can lead to a substantial improvement in performance.

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